Universität Tübingen, Fachbereich Mathematik
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Group Representations in Physics

Homework Assignment 4 (due on 22 Nov 2017)

Problem 18
Three spin-$\frac{1}{2}$ particles define a representation D of S_3 on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^8$ by permutations of the particles, i.e. e.g. $D\left((12)\right)|{\uparrow\downarrow\uparrow}\rangle = |{\downarrow\uparrow\uparrow}\rangle$.

Which irreducible representations of S_3 are contained in D and how often does each of them appear?

Problem 19
Let $g = \begin{pmatrix} u & -\overline{v} \\ v & \overline{u} \end{pmatrix}$, $u, v \in \mathbb{C}$ with $|u|^2 + |v|^2 = 1$.

a) Verify that $g \in \mathrm{SU}(2)$, and explain why every $g \in \mathrm{SU}(2)$ can be written in this way.

The basis vectors $|{\uparrow}\rangle$ and $|{\downarrow}\rangle$ of $\mathbb{C}^2$, as defined in the lecture, transform in the two-dimensional representation $\Gamma^2(g) = g \ \forall\, g \in \mathrm{SU}(2)$.

b) Write $\Gamma^2(g)|{\uparrow}\rangle$ and $\Gamma^2(g)|{\downarrow}\rangle$ as linear combinations of $|{\uparrow}\rangle$ and $|{\downarrow}\rangle$.

Consider now $\mathbb{C}^2 \otimes \mathbb{C}^2$ with the product basis $|{\uparrow\uparrow}\rangle = |{\uparrow}\rangle \otimes |{\uparrow}\rangle$ etc. (cf. lecture). Under $\mathrm{SU}(2)$ this basis transforms in $\Gamma^{2\otimes 2} = \Gamma^2 \otimes \Gamma^2$.

c) Expand $\Gamma^{2\otimes 2}|{\uparrow\uparrow}\rangle$ etc. in the product basis.

d) Show: $\mathrm{span}(|0,0\rangle)$ and $\mathrm{span}(|1,1\rangle, |1,0\rangle, |1,-1\rangle)$ (as defined in the lecture) are invariant under $\mathrm{SU}(2)$, and thus carry one- and three-dimensional representations of $\mathrm{SU}(2)$, respectively, i.e. $\Gamma^{2\otimes 2} = \Gamma^1 \oplus \Gamma^3$.

e) Explicitly determine the representation matrices $\Gamma^1(g)$ and $\Gamma^3(g)$.

On $\mathbb{C}^2 \otimes \mathbb{C}^2$ also acts – as in Problem 18 – a representation D of $S_2 \cong \mathbb{Z}_2 = \{e, (12)\}$.

f) In which representations of S_2 do the vectors $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$ and $|0,0\rangle$ transform?

Problem 20
We consider a rotationally invariant Hamiltonian. Let E be an eigenvalue of H with eigenspace V_E spanned by the spherical harmonics $Y_{1m}(\varphi, \vartheta) = \cos\vartheta \mathrm{e}^{\mathrm{i}m\varphi}$ with a fixed radial part R, i.e. $V_E = \mathrm{span}(\{R(r)Y_{1m}(\varphi, \vartheta) : m = -1, 0, 1\})$.[1]

V_E carries a three-dimensional irreducible representation of $\mathrm{O}(3)$, defined by $(\Gamma(U)\psi)(x) = \psi(U^{-1}x)$. $\mathrm{O}(3)$ contains the subgroup $D_3 = \{e, C, \bar{C}, \sigma_1, \sigma_2, \sigma_3\} \cong S_3$, where C and $\bar{C}$ denote rotations about the z-axis (cf. Section 2.4.1).

Study the effect of perturbations that are only invariant under D_3 or $\mathbb{Z}_3 \cong \{e, C, \bar{C}\}$. Determine the relevant irreducible representations with their multiplicities and sketch the possible splitting of energy levels.

Problem 21
Let V be a finite-dimensional vector space and $P : V \to V$ a linear operator with $P^2 = P$.

a) Show that there exist subspaces U and W with $V = U \oplus W$, $P|_U = \mathbb{1}$ and $P|_W = 0$.

Let $\langle \cdot, \cdot \rangle$ be a scalar product on V, and let $P^\dagger = P$.

b) Show that $U = W^\perp$.

[1] We use spherical coordinates
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\vartheta\cos\varphi \\ r\sin\vartheta\sin\varphi \\ r\cos\vartheta \end{pmatrix}.$$