Universität Tübingen, Fachbereich Mathematik Wintersemester 17/18
Dr. Stefan Keppeler & Nicolai Rothe

Group Representations in Physics

Homework Assignment 5 (due on 29 Nov 2017)

Problem 22
We consider once more the CO_2 molecule of Problem 15.

a) How many non-equivalent irreps does the symmetry group V_4 have, and what are their dimensions?

b) Determine the character table for V_4.

In Problem 15 we found a six-dimensional representation of V_4.

c) Which irreps are contained in this six-dimensional representation?

d) Decompose the six-dimensional carrier space into irreducible invariant subspaces by applying the generalised projection operators.

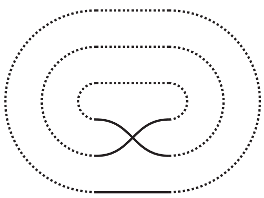
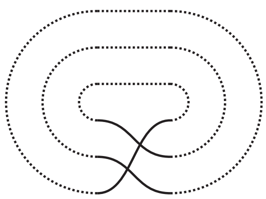
Problem 23
$V = \mathbb{C}^2$ carries the 2-dimensional irreducible representation of $D_3 \cong S_3$ (cf. Section 2.4.1). On $W = V \otimes V$ we consider the corresponding product representation. Decompose W into irreducible invariant subspaces by applying the generalised projection operators, and determine the Clebsch-Gordan coefficients.

Problem 24
For $\sigma \in S_n$ and $j = 1, \ldots, n$ let $k_j(\sigma)$ be the number of (disjoint) cycles of length j in σ, e.g. $k_1(e) = n$ and $k_j(e) = 0 \; \forall \, j > 1$. Show:

a) The conjugacy class of σ is determined by its cycle structure, i.e.

$$[\sigma] := \{\tau\sigma\tau^{-1} : \tau \in S_n\} = \{\tau \in S_n : k_j(\tau) = k_j(\sigma), j = 1, \ldots, n\}\,.$$

Can you come up with a proof (or illustration) of this fact in birdtrack notation (see Section 1.4)?
Hint: In order to make the cycle strcture visible consider the birdrack diagram of σ and connect the first line on the left to the first line on the right etc.; e.g. for $(12), (132) \in S_3$ consider


and

.

b) The number of elements of a class is given by

$$|[\sigma]| = \frac{n!}{\prod\limits_{j \le n} k_j! j^{k_j}}\,.$$

Problem 25

For $A \in \mathbb{C}^{n\times n}$ the matrix exponential is defined as

$$\mathrm{e}^A = \exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} \,.$$

Prove:

a) The series converges absolutely and uniformly.
HINT: On $\mathbb{C}^{n\times n}$ use the operator norm

$$\|A\| = \sup_{v\in\mathbb{C}^n\setminus\{0\}} \frac{|Av|}{|v|} \,,$$

for which we have $\|AB\| \leq \|A\| \, \|B\|$.

b) For $T \in \mathrm{GL}(n)$ we have $\mathrm{e}^{TAT^{-1}} = T\mathrm{e}^A T^{-1}$.

c) e^{tA} is the unique solution of the initial value problem $\dot{X}(t) = AX(t)$, $X(0) = \mathbb{1}$.

d) For $t, s \in \mathbb{C}$ we have $\mathrm{e}^{(t+s)A} = \mathrm{e}^{tA}\mathrm{e}^{sA}$.

e) $\left(\mathrm{e}^A\right)^\dagger = \mathrm{e}^{(A^\dagger)}$.

f) $\det \mathrm{e}^A = \mathrm{e}^{\operatorname{tr} A}$.