Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler & Nicolai Rothe Wintersemester 17/18

Group Representations in Physics

Homework Assignment 6 (due on 6 Dec 2017)

Problem 26
We consider the abelian group $C_3 = \{e, a, a^{-1}\} \cong \mathbb{Z}_3$.

a) How many (non-equivalent) irreps does C_3 have, what are their dimensions and how often do they appear in the regular rep?

b) Show that
$$e_1 = \frac{1}{3}(e + a + a^{-1})$$
is a primitive idempotent, generating the trivial rep.

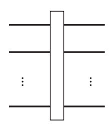
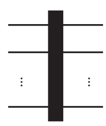
c) Use the ansatz
$$e_2 = xe + ya + za^{-1}$$
in order to find all primitive idempotents.

d) For each primitive idempotent find out whether it generates a new (non-equivalent) irrep or an irrep equivalent to one generated by a previous idempotent.

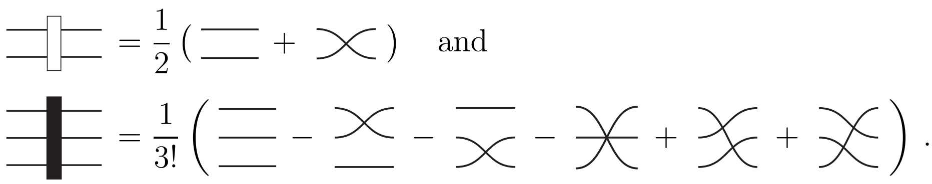
e) Specify all minimal left ideals and construct the corresponding irreps of C_3. Collect your results in a table.

Problem 27
Show that the symmetriser $s = \sum_p p$ and the anti-symmetriser $a = \sum_p \mathrm{sgn}(p)p$ for S_n are essentially idempotent and primitive. (Lemma 15)

Problem 28
In birdtrack notation we denote symmetrisers and anti-symmetrisers by open and solid bars, respectively, i.e.

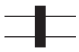
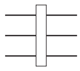
$\frac{1}{n!}s =$  and $\frac{1}{n!}a =$  .

Note that we include a factor of $\frac{1}{n!}$ in the definition of bars over n lines. For instance,

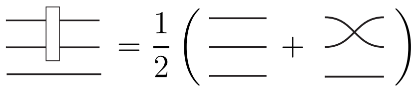


(*)

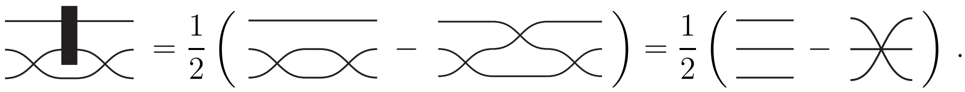
Notice that in birdtrack notation the sign of a permutation, $(-1)^K$, is determined by the number K of line crossings; if more than two lines cross in a point, one should slightly perturb the diagram before counting, e.g.  $\rightsquigarrow$  (K=3).

a) Expand  and  as in (∗).

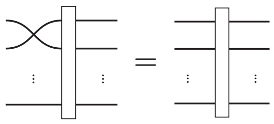
We also use the corresponding notation for partial (anti-)symmetrisation over a subset of lines, e.g.



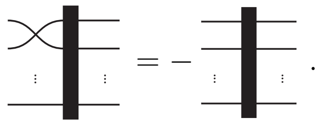
or



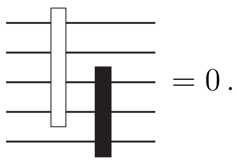
It follows directly from the definition of S and A that when intertwining any two lines S remains invariant and A changes by a factor of (-1), i.e.



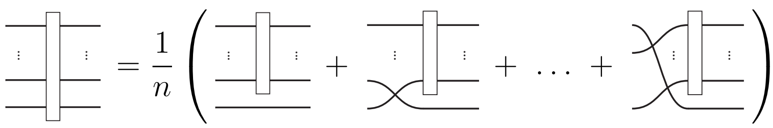
and

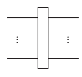


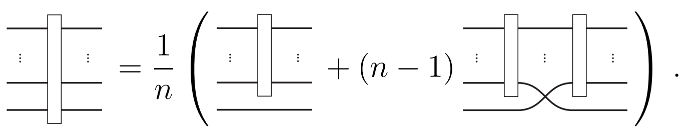
b) Explain why this implies that whenever two (or more) lines connect a symmetriser to an anti-symmetrizer the whole expression vanishes, e.g.



Symmetrisers and anti-symmetrisers can by built recursively. To this end notice that on the r.h.s. of



we have sorted the terms according to where the last line is mapped – to the nth, to the $(n-1)$th, ..., to the first line line. Multiplying with  from the left and disentangling lines we obtain the compact relation



c) Derive the corresponding recursion relation for anti-symmetrisers.