Universität Tübingen, Fachbereich Mathematik Wintersemester 17/18
Dr. Stefan Keppeler & Nicolai Rothe

Group Representations in Physics

Homework Assignment 7 (due on 13 Dec 2017)

Problem 29
Verify that the Young operators for the standard Young tableaux for S_3 (see Section 5.3) are essentially idempotent and primitive and add up to the identity. (Use cycle and birdtrack notation for some of your calculations.) Determine once more the characters of the irreps of S_3 by using the methods of Section 4.3.1.

Problem 30
Let V be a vector space and $A : V \to V$ a linear map. Show that if A is nilpotent (i.e. if for some $n \in \mathbb{N}$ we have $A^n v = 0 \; \forall \, v \in V$) then $\operatorname{tr} A = 0$.

Problem 31
For a fixed partition λ of $n \in \mathbb{N}$ we define an ordering of standard tableaux as follows. Consider the sequence $(r^p_\lambda)_j$, $j = 1, \ldots, n$, of numbers in the boxes of Θ^p_λ starting with the first row read from left to right, then the second row from left to right etc. We say that $\Theta^p_\lambda > \Theta^q_\lambda$ if the first non-vanishing term in the sequence $(r^p_\lambda)_j - (r^q_\lambda)_j$, $j = 1, \ldots, n$, is positive. Then, e.g., the standard tableaux for $\lambda = (3, 2)$ are ordered as

$$\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5 \\ \cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&4\\ \hline 3&5 \\ \cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1&2&5\\ \hline 3&4 \\ \cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5 \\ \cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1&3&5\\ \hline 2&4 \\ \cline{1-2}\end{array} .$$

a) Prove that $e^p_\lambda e^q_\lambda = 0$ if $\Theta^p_\lambda > \Theta^q_\lambda$.

b) Show that (a) implies that the left ideals generated by the standard tablaux for a fixed partition are linearly independent.

Problem 32
Determine the dimensions of all irreps of S_4 using the methods of Section 5.5.

Problem 33

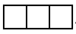
In the quark model baryons are made out of three quarks. The latter are characterised i.a. by the quantum numbers I (isospin) and Y (hyper charge).

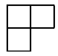
We have $(I, Y) = \left(\frac{1}{2}, \frac{1}{3}\right)$ for the up-quark, $|u\rangle$, $(I, Y) = \left(-\frac{1}{2}, \frac{1}{3}\right)$ for the down-quark, $|d\rangle$, and $(I, Y) = \left(0, -\frac{2}{3}\right)$ for the strange-quark, $|s\rangle$. For products like $|udd\rangle = |u\rangle \otimes |d\rangle \otimes |d\rangle$ the values of I and Y are given by the sums of the values for the individual quarks.

For combinations of 3 quarks we thus have a 27-dimensional space V, which carries a representation of S_3 (by permutation of the factors).

a) Which irreps are contained in this representation and what are their multiplicities?

b) Let $U \subset V$ be an irreducible invariant subspace. What can we say about the values of I and Y on U?

c) In a (I, Y)-diagram mark all points corresponding to vectors transforming in the irrep defined by .

d) Repeat part (c) for the irrep with Young diagram . You find some potentially useful `Octave`/MATLAB-Code on the course webpage.

Problem 34

Let $\vec{n} \in S^2 \hookrightarrow \mathbb{R}^3$ be a unit vector in $\mathbb{R}^3$ and $\varphi \in \mathbb{R}$. We denote by σ_i, $i = 1, 2, 3$ the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and we define

$$\vec{\sigma} = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix}.$$

Show that

$$\exp\left(-\mathrm{i}\tfrac{\varphi}{2}\vec{\sigma}\cdot\vec{n}\right) = \mathbb{1}\cos\tfrac{\varphi}{2} - \mathrm{i}\vec{\sigma}\cdot x \sin\tfrac{\varphi}{2},$$

and verify that $\exp\left(-\mathrm{i}\tfrac{\varphi}{2}\vec{\sigma}\cdot\vec{n}\right) \in \mathrm{SU}(2)$.

HINT: First calculate $(\vec{\sigma}\cdot\vec{n})^2$.