Universität Tübingen, Fachbereich Mathematik — Wintersemester 17/18

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Group Representations in Physics

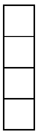
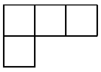
Homework Assignment 8 (due on 20 Dec 2017)

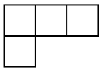
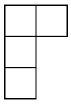
Problem 35

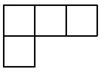
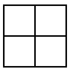
Determine the character table of S_4 using the methods of Section 5.5 (see also Problem 32).

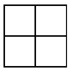
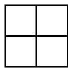
Problem 36

Consider the following product representations of S_4, determine which irreps they contain and how many times. Also write their decomposition into irreps in terms of Young diagrams.

a)  $\otimes$ 

b)  $\otimes$ 

c)  $\otimes$ 

d)  $\otimes$ 

Problem 37

Let G be a connected Lie group and H a totally disconnected normal subgroup. Show that $gh = hg$ for all $g \in G$ and for all $h \in H$.

Problem 38

The so-called Lie algebra of SU(2) is the (real) vector space

$$\mathfrak{su}(2) = \{X \in \mathbb{C}^{2\times 2} : \operatorname{tr}(X) = 0, X^\dagger = X\}.$$

A basis is given by the Pauli matrices (see Problem 34). Show:

a) SU(2) acts on $\mathfrak{su}(2)$ by conjugation: $X \mapsto UXU^\dagger$.

b) $\langle X, Y\rangle := \frac{1}{2}\operatorname{tr}(XY)$ defines a scalar product on $\mathfrak{su}(2)$.
Hint: Begin by calculating $\operatorname{tr}(\sigma_i\sigma_j)$.

c) Every $U \in \mathrm{SU}(2) \cong S^3$ (cf. Problem 19) can be written as $\mathrm{e}^{-\frac{1}{2}\mathrm{i}\alpha\vec{\sigma}\cdot\vec{n}}$ with $\vec{n} \in S^2 \hookrightarrow \mathbb{R}^3$ (cf. Problem 34). Over which values does α run?

Problem 39

We define $\mathfrak{sl}(2,\mathbb{C}) := \{A \in \mathbb{C}^{2\times 2} : \operatorname{tr} A = 0\}$. Then the matrix exponential (cf. Problem 25) is a map

$$\exp : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{SL}(2,\mathbb{C}) = \{B \in \mathbb{C}^{2\times 2} : \det B = 1\}.$$

a) Show that the matrix

$$S_a = \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$$

is in the image of exp iff $a = 0$.

b) Is $\mathrm{SL}(2,\mathbb{C})$ compact?