Universität Tübingen, Fachbereich Mathematik
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Group Representations in Physics

Homework Assignment 9 (due on 10 Jan 2018)

Problem 40

The elements of $\mathfrak{su}(2)$ can be written as $X = \vec{\sigma}\cdot\vec{x}$ with $\vec{x} \in \mathbb{R}^3$ (cf. Problems 34 & 38). The action of SU(2) on $\mathfrak{su}(2)$ by conjugation (see Problem 38) then defines a homomorphism

$$\varphi : \mathrm{SU}(2) \to \mathrm{GL}(3,\mathbb{R})$$
$$\vec{\sigma}\cdot\varphi(U)\vec{x} := U(\vec{\sigma}\cdot\vec{x})U^\dagger\,.$$

Show that

a) $\varphi(U)_{ij} = \frac{1}{2}\,\mathrm{tr}(\sigma_i U \sigma_j U^\dagger)$,

b) $\varphi(U)^T = \varphi(U)^{-1}$, and

c) $\det(\varphi(U)) = 1$. Hint: Recall the connectedness properties of SO(3).

Hence $\varphi(\mathrm{SU}(2)) \subset \mathrm{SO}(3)$.

d) Determine the kernel of φ.

e) Calculate $\varphi(U_\alpha)$ for $U_\alpha = \mathrm{e}^{-\frac{1}{2}\mathrm{i}\alpha\sigma_3}$, $\alpha \in [0, 4\pi)$ and explain that $\varphi(\mathrm{SU}(2)) = \mathrm{SO}(3)$. What can we now conclude using the homomorphism theorem (Problem 9)?

Problem 41

a) Determine the Haar measure for SU(2) in axis-angle parametrisation,

$$U = \exp\left(-\mathrm{i}\tfrac{\alpha}{2}\vec{\sigma}\cdot\vec{x}\right)\,,$$

with $0 \le \alpha < 4\pi$ and $\vec{x} \in S^2 \hookrightarrow \mathbb{R}^3$. Normalise s.t. vol(SU(2)) = 1.

Hint: It is convenient to first show $(\vec{x}\cdot\vec{\sigma})(\vec{y}\cdot\vec{\sigma}) = \mathbb{1}\vec{x}\cdot\vec{y} + \mathrm{i}\vec{\sigma}(\vec{x}\times\vec{y})$ and to use the unit vectors $\vec{e}_r, \vec{e}_\varphi, \vec{e}_\vartheta$ for spherical coordinates.

b) Use the result of (a) together with Problem 37 in order to determine the Haar measure for SO(3) in the axis-angle parametrisation.

Problem 42

The orthogonal group $\mathrm{O}(n)$ is a submanifold of $\mathbb{R}^{n\times n}$. It is given by the level set to the regular value 0 of the function $F : \mathbb{R}^{n\times n} \to \mathrm{Sym}(n)$, $F(A) = A^T A - \mathbb{1}$, i.e. $\mathrm{O}(n) = \{A \in \mathbb{R}^{n\times n} : F(A) = 0\} = F^{-1}(0)$. The Lie algebra $\mathfrak{o}(n)$ is the tangent space at the identity and given by $\mathfrak{o}(n) = \ker(DF|_{\mathbb{1}})$.

Calculate $\mathfrak{o}(n)$ explicitly and determine the dimension of $\mathrm{O}(n)$.