Universität Tübingen, Fachbereich Mathematik Wintersemester 17/18
Dr. Stefan Keppeler & Nicolai Rothe

Group Representations in Physics

Homework Assignment 10 (due on 17 Jan 2018)

Problem 43

a) Prove the following formula for the characters of the irreps Γ^j of SU(2),

$$\chi^j(\alpha) = \frac{\sin\left((2j+1)\alpha/2\right)}{\sin(\alpha/2)}\,.$$

b) Consider the product representation $\Gamma^j \otimes \Gamma^k$ with $j \geq k$. Show that every irrep Γ^ℓ with $\ell = j-k, \ldots, j+k$ appears exactly once in the decomposition of $\Gamma^j \otimes \Gamma^k$, and that all other irreps are absent.

Problem 44

Let V be a (complex, finite dimensional) vector space and let V^* be its dual, i.e. the space of all linear maps $V \to \mathbb{C}$. For a linear map $A : V \to V$ we define its dual $A^* : V^* \to V^*$ by $V^* \ni f \mapsto A^*(f) := f \circ A$. Let G be group and $\Gamma : G \to \mathrm{GL}(V)$ a representation.

a) Define a representation $\Gamma^* : G \to \mathrm{GL}(V^*)$ in a natural way.

HINT: Simply replacing $\Gamma(g) : V \to V$ by its dual map doesn't quite work (why?) but with a slight modification it does.

Let $\{e_j\}$ be a basis of V and $\{f_j\}$ the corresponding dual basis, i.e. $f_j(e_k) = \delta_{jk}\ \forall\, j,k = 1,\ldots,\dim V = \dim V^*$. For $g \in G$ we express $\Gamma(g) : V \to V$ and $\Gamma^*(g) : V^* \to V^*$ as matrices in the bases $\{e_j\}$ and $\{f_j\}$, respectively.

b) What is the relation between these two matrices? What happens if Γ is unitary?

Problem 45

Let Γ by an irrep of SU(2) on $\mathbb{C}^n$. Show that there exists a $T \in \mathrm{GL}(n,\mathbb{C})$ with $T^2 \in \{\pm\mathbb{1}\}$ s.t.

$$\overline{\Gamma(g)} = T\Gamma(g)T^{-1} \quad \forall\, g \in \mathrm{SU}(2).$$

In which cases do we have $T^2 = \mathbb{1}$?

HINT: First find T for the defining representation. Observe that $T \in \mathrm{SU}(2)$ and then investigate $\Gamma(T)$.

Problem 46

Let $A, B \in \mathbb{C}^{n\times n}$. We define

$$\mathrm{ad}_A B := [A, B] \,.$$

a) Show (for $t \in \mathbb{R}$)

$$\begin{aligned} \mathrm{e}^{tA} B \mathrm{e}^{-tA} &= \mathrm{e}^{t\mathrm{ad}_A} B \\ &= B + t[A, B] + \frac{t^2}{2}[A, [A, B]] + \ldots \end{aligned}$$

HINT: Show that both sides solve the same initial value problem.

b) Let $Z : \mathbb{R} \to \mathbb{C}^{n\times n}$ be analytic; $\beta, t \in \mathbb{R}$. Show

$$\frac{\partial}{\partial t} \mathrm{e}^{\beta Z(t)} = \int_0^\beta \mathrm{e}^{(\beta-u)Z(t)} Z'(t) \mathrm{e}^{uZ(t)} du \,.$$

HINT: Show that both sides (as functions of β) solve the same initial value problem.

c) We set $\mathrm{e}^{Z(t)} := \mathrm{e}^{tA}\mathrm{e}^{tB}$. In the identity of part (b) choose $\beta = 1$ and $u = 1 - x$, and conclude that

$$\int_0^1 \mathrm{e}^{xZ(t)} Z'(t) \mathrm{e}^{-xZ(t)} dx = A + \mathrm{e}^{t\mathrm{ad}_A} B \,.$$

Under which conditions do we have $Z(t) = t(A + B)$?

d) Expand Z of part (c) about $t = 0$ and show the Baker-Campbell-Hausdorff formula

$$Z(t) = t(A + B) + \frac{t^2}{2}[A, B] + \frac{t^3}{12}\left([A, [A, B]] + [[A, B], B]\right) + \mathcal{O}(t^4) \,.$$

HINT: Calculate the l.h.s. of (c) using (a), and compare coefficients of power series.