Universität Tübingen, Fachbereich Mathematik
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Group Representations in Physics

Homework Assignment 11 (due on 24 Jan 2018)

Problem 47
Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let ad be the adjoint representation of $\mathfrak{g}$, i.e. $\mathrm{ad}_X(Y) = [X, Y]$. The map

$$\begin{aligned} K : \mathfrak{g} \times \mathfrak{g} &\to \mathbb{R} \\ (X, Y) &\mapsto K(X, Y) := \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) \end{aligned}$$

is called Killing-Form.

Show:

a) K is bilinear and symmetric.

b) $K\big(\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)\big) = K(X, Y) \quad \forall\, X, Y \in \mathfrak{g}$ and $\forall\, g \in G$.

For semi-simple Lie groups (which we haven't defined, but the classical groups $\mathrm{SU}(n)$ und $\mathrm{SO}(n)$ are examples) K is positive definite, i.e. it defines a scalar product. In this case we choose an orthonormal basis $\{X_j\}$ with respect to K, i.e. $K(X_j, X_k) = \delta_{jk}$, and define $C_2 \in E(\mathfrak{g})$ by

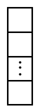
$$C_2 := \sum_j X_j X_j\,.$$

Show:

c) C_2 is independent of the choice of basis.

d) C_2 is a Casimir operator (the so-called *quadratic Casimir operator*), i.e.

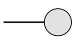
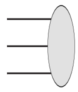
$$\mathrm{Ad}_g(C_2) = C_2 \quad \forall\, g \in G\,.$$

Problem 48
Show that the $\mathrm{GL}(N)$ irrep corresponding to the Young diagram $\Theta_a =$  with N rows is given by the determinant.

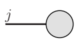
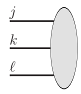
HINT: Convince yourself that for the vectors $|i_1, \ldots, i_N\rangle$ contributing to $e_a g|\alpha\rangle$ all i_k are different. Write these vectors as $p|1, \ldots, N\rangle$ with a permutation p and calculate $e_a g|1, \ldots, N\rangle$.

Problem 49

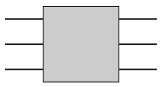
In birdtrack notation we write vectors $v \in V$ or tensors $t \in V^{\otimes n}$, as blobs with one or several lines attached, e.g.

$v =$  $\in V$ or $t =$  $\in V^{\otimes 3}$.

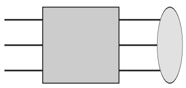
Expressed in a basis, components are

$v_j =$  or $t_{jk\ell} =$  ,

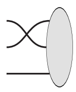
i.e. we write indices on the lines. Linear maps $A : V^{\otimes n} \to V^{\otimes n}$ are represented by blobs with n legs on each side, e.g.

$A =$  $: V^{\otimes 3} \to V^{\otimes 3}$,

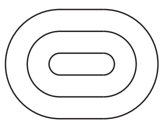
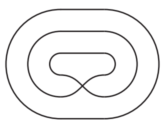
and with $t \in V^{\otimes 3}$ we have

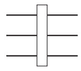
$At =$  $\in V^{\otimes 3}$.

Birdtracks for permutations $p \in S_n$, or $\in \mathcal{A}(S_n)$, see Problems 24 & 28, can now be applied (as linear maps) to elements of $V^{\otimes n}$, e.g.

$(12)t =$  .

We obtain traces of linear maps by connecting the first line on the left to the first line on the right etc. (cf. also Problem 24), with each loop contributing a factor of $\dim V = N$ (why?), e.g. for $e, (12) \in \mathcal{A}(S_3)$ we get

$\operatorname{tr} e =$  $= N^3$ and $\operatorname{tr}(12) =$  $= N^2$

a) Calculate the trace of $(132) \in S_3$ and  $\in \mathcal{A}(S_3)$.

b) Normalise the Young operators $e_1, e_3, e_2, e_2^{(23)} \in \mathcal{A}(S_3)$ of Section 5.3 s.t. they are idempotent (see Problem 29). Determine the traces of these primitive idempotents. These are the dimensions of the $\mathrm{GL}(N)$ irreps contained in $V^{\otimes 3}$.

Hint: Some identities from Problem 28 are useful.