Quantum spectral properties of graphs with spin

J. Harrison and J. Bolte - Universität Ulm

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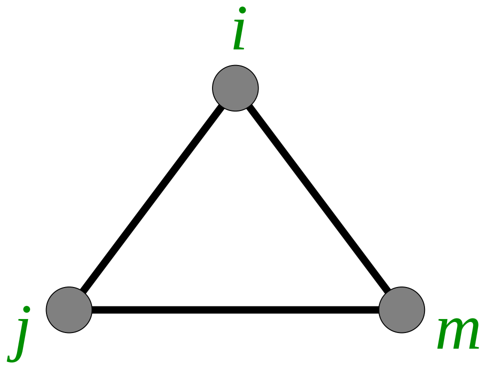
Graphs



V vertices

B bonds

(ij) labels transition from i to j along bond $\{ij\}$

$\sigma_{(ij)(mi)}$	Transition amplitude from (mi) to (ij)
$\Sigma^{(i)}$	Unitary vertex scattering matrix
$\lvert\sigma_{(ij)(mi)}\rvert^2$	Classical transition probability
$\mathbb{S}$	Graph scattering matrix
$\dim(\mathbb{S})$	$2B$

$$\mathbb{S}_{(ij)(kl)} := \delta_{il}\sigma_{(ij)(ki)}\mathrm{e}^{\mathrm{i}\phi_{\{kl\}}} \tag{1}$$

Classical dynamics

- Stochastic matrix of transition probabilities M.

$$\rho_{n+1} = M\rho_n \tag{2}$$

where $\rho = (\rho_1, \ldots, \rho_{2B})^T$

- Quantum system defined by unitary $\mathbb{S}$.

$$M_{(ij)(ki)} = \left|\mathbb{S}_{(ij)(ki)}\right|^2 \tag{3}$$

(M doubly stochastic - rows and columns sum to 1)

- **Mixing:**

$$\lim_{n\to\infty} M^n \rho = \frac{1}{2B}(1, \ldots, 1)^T \quad \text{or} \quad \lim_{n\to\infty} (M^n)_{bc} = \frac{1}{2B} \tag{4}$$

Quantum graphs

1d time independent Schrödinger eqn.

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) = E\psi(x) \tag{5}$$

1d time independent Dirac eqn.

$$\left(-\mathrm{i}\hbar c\,\alpha\,\frac{\partial}{\partial x} + mc^2\,\beta\right)\psi(x) = E\psi(x) \tag{6}$$

$$\alpha^2 = \beta^2 = \mathrm{I} \quad \alpha\beta + \beta\alpha = 0$$

$$\alpha = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hilbert space

$$\mathcal{H} := \bigoplus_{b=1}^{B} L^2([0, L_b]) \quad \left(\otimes\mathbb{C}^2\right) \tag{7}$$

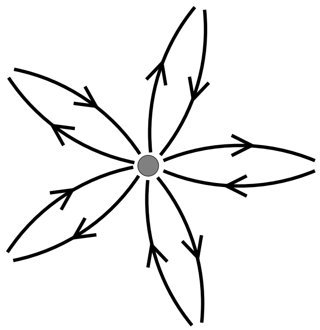
Eigenspinors *(Dirac op.)*

+ve energy solns. of 1d time independent Dirac eqn.

$$\psi_k(x) = \mu(k) \begin{pmatrix} 1 \\ \mathrm{i}\gamma(k) \end{pmatrix} \mathrm{e}^{\mathrm{i}kx} + \hat{\mu}(k) \begin{pmatrix} 1 \\ -\mathrm{i}\gamma(k) \end{pmatrix} \mathrm{e}^{-\mathrm{i}kx} \tag{8}$$

$$\gamma(k) := \frac{E - mc^2}{\hbar ck} \qquad E = \sqrt{(\hbar ck)^2 + m^2c^4} \tag{9}$$

Consider isolated vertex with paired bonds ($m = 0$).



$$\psi_k^b(x_b) = \mu_\alpha^b \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \mathrm{e}^{\mathrm{i}kx_b} + \mu_\beta^b \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \mathrm{e}^{\mathrm{i}kx_b} + \hat{\mu}_\alpha^b \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \mathrm{e}^{-\mathrm{i}kx_b} + \hat{\mu}_\beta^b \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \mathrm{e}^{-\mathrm{i}kx_b} \tag{10}$$

S-matrix for the Dirac op.

2 lin. indep. eigenspinors per transition, $\dim(\mathbb{S}) = 4B$.
Divide $\mathbb{S}$ into 2×2 blocks $\mathbb{S}^{(ij)(kl)}$.

$$\mathbb{S}_{(ij)(kl)} := \delta_{il}\, \sigma_{(ij)(ki)}\, u^{(ij)(ki)}\, \mathrm{e}^{\mathrm{i}\phi_{\{kl\}}} \tag{11}$$

$u^{(ij)(ki)} \in \mathrm{SU}(2)$

Time-reversal symmetry $\implies$

$$\mathbb{S}^{(lk)(ji)} = \left|\mathbb{S}^{(ij)(kl)}\right| \left(\mathbb{S}^{(ij)(kl)}\right)^{-1} \tag{12}$$

To satisfy (12) take $\Sigma^{(i)}$ symmetric and

$$u^{(ij)(ki)} := u_j^{(i)} \left(u_k^{(i)}\right)^{-1} \tag{13}$$

(Schrödinger op.: time-reversal symmetry implies $\Sigma^{(i)}$ symmetric)

The form factor

U unitary matrix, eigenvalues $\mathrm{e}^{\mathrm{i}\phi_1}, \ldots, \mathrm{e}^{\mathrm{i}\phi_N}$.

Level density

$$\rho(\phi) := \frac{1}{N}\sum_{i=1}^{N}\delta(\phi-\phi_i) = \frac{1}{2\pi N}\sum_{i=1}^{N}\sum_{n=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}n(\phi-\phi_i)} \tag{14}$$

$$= \frac{1}{2\pi N}\sum_{n=-\infty}^{\infty}(\operatorname{tr} U^n)\mathrm{e}^{\mathrm{i}n\phi} \tag{15}$$

Two-point correlation function

$$R_2(\Delta\phi) = \langle\rho(\phi)\rho(\phi+\Delta\phi)\rangle_{ensemble} \tag{16}$$

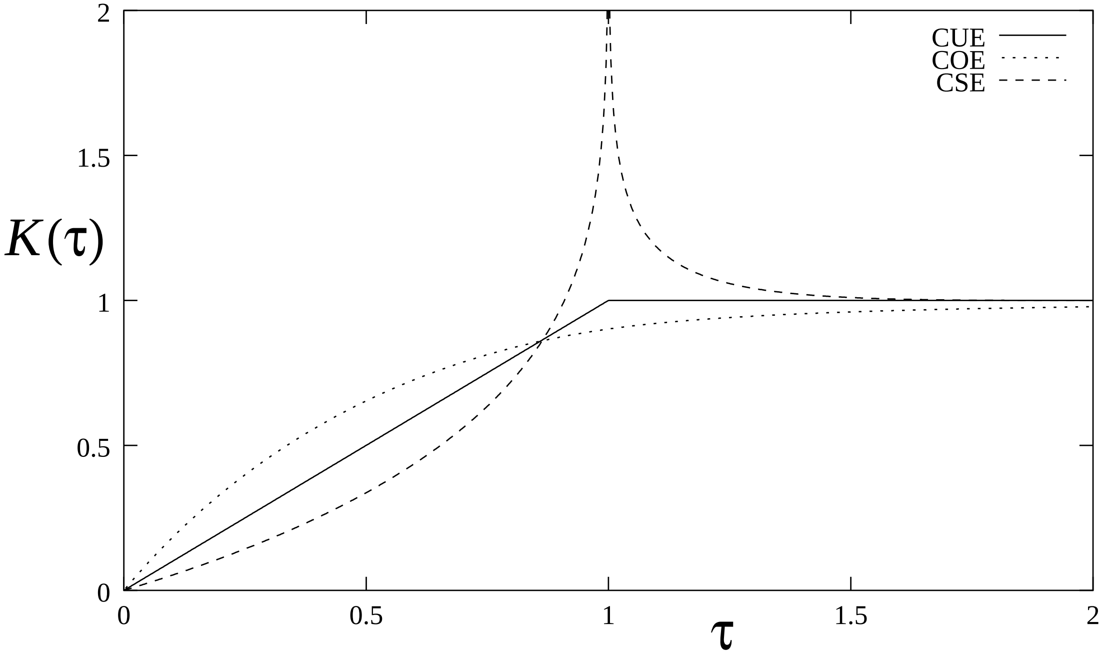
Form factor: Fourier transform of $R_2(\phi)$

$$K(\tau) := \frac{1}{N}\left\langle|\operatorname{tr} U^n|^2\right\rangle_{ensemble} \qquad \tau = \frac{n}{N} \tag{17}$$

Conjecture -Bohigas, Giannoni, Schmit

The eigenvalue statistics of a quantum system whose classical analogue is chaotic correspond to those of random matrices.

GUE/CUE	no time-reversal symmetry
GOE/COE	time-reversal symmetry, integer spin
GSE/CSE	time-reversal symmetry, half-integer spin



Form factor of a quantum graph

Schrödinger eqn.

Define ensemble of S-matrices

$$\mathbb{S}_{\phi}^{(ij)(kl)} := \delta_{il}\sigma_{(ij)(ki)}\mathrm{e}^{\mathrm{i}\phi_{kl}} \tag{18}$$

$$\mathrm{tr}\,\mathbb{S}_{\phi}^{n} = \sum_{p\in P_n} \frac{n}{r_p} A_p\,\mathrm{e}^{\mathrm{i}\pi\mu_p}\mathrm{e}^{\mathrm{i}\phi_p} \tag{19}$$

where $p = (b_1, b_2, \ldots b_n)$ is a periodic orbit.

$$\begin{aligned} A_p\,\mathrm{e}^{\mathrm{i}\pi\mu_p} &:= \sigma_{b_n b_{n-1}}\sigma_{b_{n-1}b_{n-2}}\ldots\sigma_{b_1 b_n} \\ \phi_p &:= \sum_{j=1}^{n}\phi_{\{b_j\}} \end{aligned} \tag{20}$$

$$K_{\mathrm{orth}}(\tau) := \frac{1}{N}\left\langle |\,\mathrm{tr}\,\mathbb{S}_{\phi}^{n}|^2\right\rangle_{\phi} = \frac{n^2}{2B}\sum_{p,q\in P_n}\frac{A_pA_q}{r_pr_q}\,\mathrm{e}^{\mathrm{i}\pi(\mu_p-\mu_q)}\,\delta_{\phi_p,\phi_q} \tag{21}$$

Dirac eqn.

$$\mathbb{S}_{\phi}^{(ij)(kl)} := \delta_{il}\, \sigma_{(ij)(ki)}\, u^{(ij)(ki)}\, \mathrm{e}^{\mathrm{i}\phi_{kl}} \tag{22}$$

$u^{(ij)(ki)} \in \mathrm{SU}(2)$ is a spin rotation at i.

$$d_p := u^{b_n b_{n-1}} u^{b_{n-1} b_{n-2}} \dots u^{b_2 b_1} \tag{23}$$

$$\mathrm{tr}\, \mathbb{S}_{\phi}^{n} = \sum_{p \in P_n} \frac{n}{r_p} A_p\, \mathrm{e}^{\mathrm{i}\pi\mu_p}\, \mathrm{tr}(d_p)\, \mathrm{e}^{\mathrm{i}\phi_p} \tag{24}$$

$$K_{\mathrm{sympl}}(\tau) := \frac{1}{4N} \left\langle |\, \mathrm{tr}\, \mathbb{S}_{\phi}^{n}|^2 \right\rangle_{\phi} \tag{25}$$

$$= \frac{n^2}{4(2B)} \sum_{p,q \in P_n} \frac{A_p A_q}{r_p r_q} \mathrm{e}^{\mathrm{i}\pi(\mu_p - \mu_q)}\, \mathrm{tr}(d_p)\, \mathrm{tr}(d_q)\, \delta_{\phi_p,\phi_q} \tag{26}$$

Random Matrix Theory

$$
\begin{aligned}
K_{\mathrm{CSE}}(\tau) &= \frac{\tau}{2} + \frac{\tau^2}{4} + \frac{\tau^3}{8} + \frac{\tau^4}{12} + \dots \\
\frac{1}{2} K_{\mathrm{COE}}\left(\frac{\tau}{2}\right) &= \frac{\tau}{2} - \frac{\tau^2}{4} + \frac{\tau^3}{8} - \frac{\tau^4}{12} + \dots
\end{aligned}
\tag{27}
$$

Let K^m correspond to the term containing τ^m then

$$
K^m_{\mathrm{CSE}}(\tau) = \left(-\frac{1}{2}\right)^{m+1} K^m_{\mathrm{COE}}(\tau) \tag{28}
$$

According to B-G-S conjecture expect

$$
K^m_{\mathrm{sympl}}(\tau) = \left(-\frac{1}{2}\right)^{m+1} K^m_{\mathrm{orth}}(\tau) \tag{29}
$$

Background

Analysing the form factor $K(\tau)$ via periodic orbits expect small τ asymptotics agree with RMT. (B-G-S conjecture)

84/85 Diagonal approximation, 2τ

- Hanny & Ozorio de Almeida / Berry

01 Orbit pairs with one (almost) self-intersection, $-2\tau^2$

- Sieber / Sieber & Richter

02 Orbit pairs with two self-intersections, $2\tau^3$

- Berkolaiko, Schanz, Whitney (quantum graphs)

Diagonal approximation

$$K_{\mathrm{orth}}(\tau) = \frac{n^2}{2B} \sum_{p,q \in P_n} \frac{A_p A_q}{r_p r_q} \, \mathrm{e}^{\mathrm{i}\pi(\mu_p - \mu_q)} \, \delta_{\phi_p, \phi_q} \tag{30}$$

$$K_{\mathrm{orth}}^{\mathrm{diag}}(\tau) := \frac{n^2}{2B} \sum_{p \in P_n} \frac{A_p^2 + A_p A_{\overline{p}}}{r_p^2} \tag{31}$$

Time-reversal symmetry; $A_{\overline{p}} = A_p$.

Consider limit $B \to \infty, n \to \infty, \tau (= n/2B) \ll 1$.

(Neglect orbits $r_p \neq 1$)

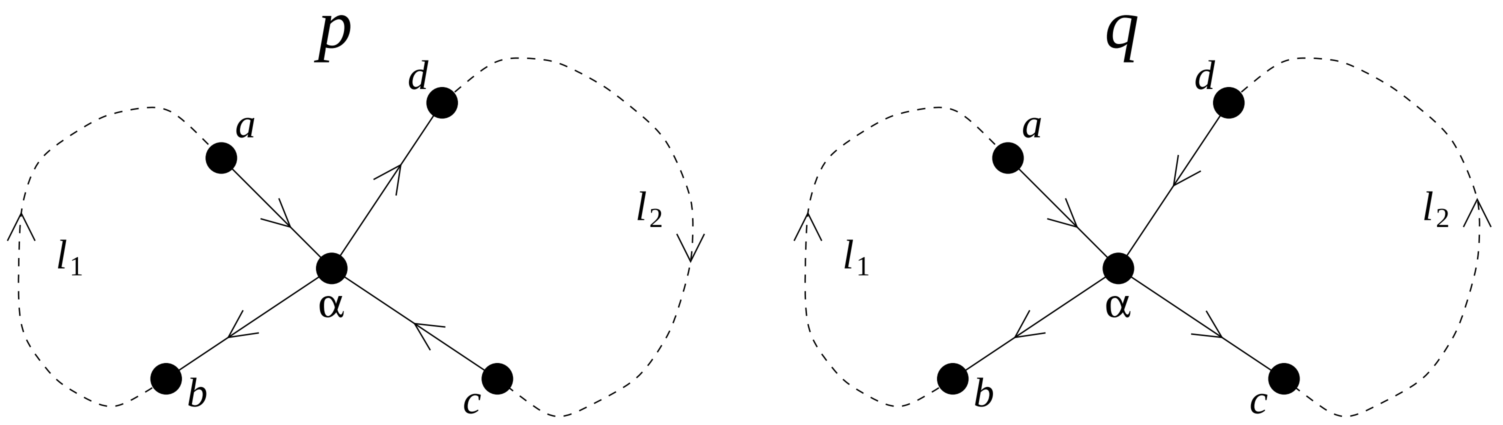
Classical Markov chain mixing,

$$\lim_{n \to \infty} n \sum_{p \in P_n} A_p^2 = \lim_{n \to \infty} \operatorname{tr} M^n = 1 \tag{32}$$

$$K_{\mathrm{orth}}^{\mathrm{diag}}(\tau) = 2\tau + o(\tau) \tag{33}$$

Diagrams

Sets of pairs of orbits related by the same pattern of permutation of arcs between self intersections and time-reversal of arcs.



D_n denotes a diagram containing orbits of length n.

Hypothesis - Diagrams reordered at $m-1$ self-intersections contribute only to K^m.

Symplectic form factor with diagrams

Fix diagram D_n with $m-1$ self-intersections.

$$K^{m,D}_{\mathrm{sympl}}(\tau) := \frac{n^2}{4(2B)} \sum_{(p,q)\in D_n} A_p A_q \,\mathrm{e}^{\mathrm{i}\pi(\mu_p-\mu_q)} \,\mathrm{tr}(d_p)\,\mathrm{tr}(d_q) \tag{34}$$

Assign elements $u_j^{(i)}$ randomly from $\Gamma \subseteq \mathrm{SU}(2)$.

$$d_p = u^{b_n b_{n-1}} \dots u^{b_2 b_1} \qquad A_p \,\mathrm{e}^{\mathrm{i}\pi\mu_p} = \sigma_{b_n b_{n-1}} \dots \sigma_{b_1 b_n}$$

d_p and A_p uncorrelated

$$K^{m,D}_{\mathrm{sympl}}(\tau) = \frac{1}{4}\left(\frac{1}{|D_n|}\sum_{(p,q)\in D_n} \mathrm{tr}(d_p)\,\mathrm{tr}(d_q)\right) \times \left(\frac{n^2}{2B}\sum_{(p,q)\in D_n} A_p A_q \,\mathrm{e}^{\mathrm{i}\pi(\mu_p-\mu_q)}\right) \tag{35}$$

Show in semiclassical limit

$$\langle \mathrm{tr}(d_p)\,\mathrm{tr}(d_q)\rangle^{m,D_n} := \frac{1}{|D_n|}\sum_{(p,q)\in D_n} \mathrm{tr}(d_p)\,\mathrm{tr}(d_q) \to \left(-\frac{1}{2}\right)^{m-1} \tag{36}$$

Subgroups $\Gamma \subseteq \mathbf{SU}(2)$

For large no. of orbits

$$\sum_{(p,q)} \to \sum_{u_{l_1} \in \Gamma} \cdots \sum_{u_{l_k} \in \Gamma} \qquad (\text{where } d_p = u_{l_1} u_{l_2} \dots u_{l_k}) \tag{37}$$

Identities:

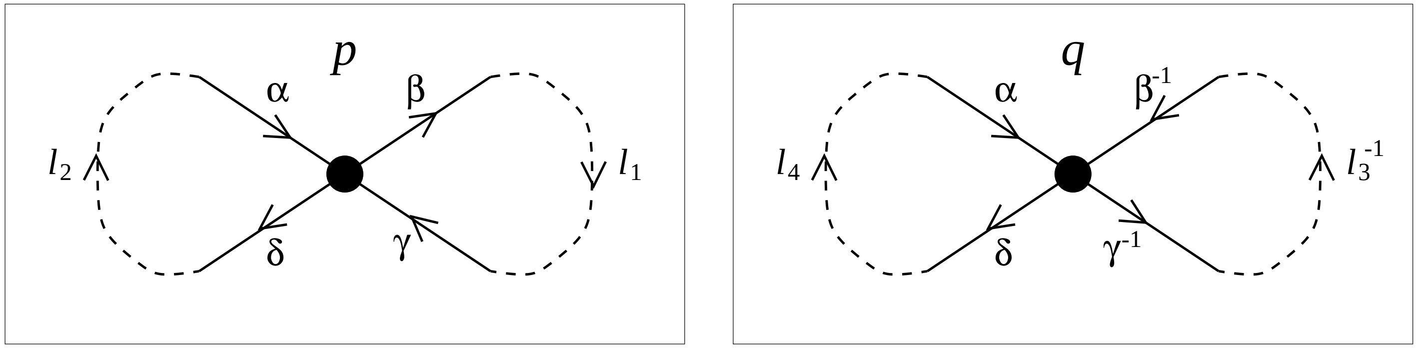
$$\frac{1}{|\Gamma|} \sum_{u \in \Gamma} \operatorname{tr}(xuyu) = -\frac{1}{2} \operatorname{tr}(xy^{-1}) \qquad (\text{quaternionic rep.}) \tag{38}$$

$$\frac{1}{|\Gamma|} \sum_{u \in \Gamma} \operatorname{tr}(xuyu^{-1}) = \frac{1}{2} \operatorname{tr}(x) \operatorname{tr}(y) \qquad (\text{irreducible rep.}) \tag{39}$$

$$\frac{1}{|\Gamma|} \sum_{u \in \Gamma} \operatorname{tr}(xu) \operatorname{tr}(yu) = \frac{1}{2} \operatorname{tr}(xy^{-1}) \qquad (\text{irreducible rep.}) \tag{40}$$

(Quaternionic rep. equivalent to its complex conjugate rep. but inequivalent to any real rep.)

Reordering at a single intersection



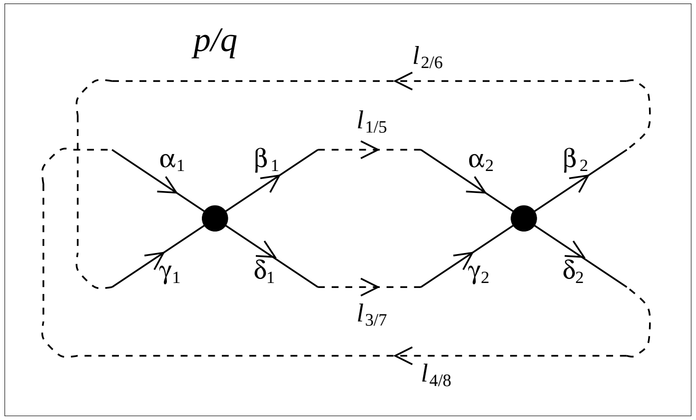
$$\langle \mathrm{tr}(d_p)\,\mathrm{tr}(d_q)\rangle = \frac{1}{|\Gamma|}\sum_{\Gamma}\cdots\frac{1}{|\Gamma|}\sum_{\Gamma}\mathrm{tr}(\alpha\beta l_1\gamma\delta l_2)\ \mathrm{tr}(\alpha\gamma^{-1}l_3^{-1}\beta^{-1}\delta l_4) \tag{41}$$

$$x := \alpha\beta \qquad y := \gamma\delta \tag{42}$$

$$\langle \ldots \rangle = \frac{1}{|\Gamma|}\sum_{\Gamma}\cdots\frac{1}{|\Gamma|}\sum_{\Gamma}\mathrm{tr}(xl_1yl_2)\ \mathrm{tr}(x(\gamma\beta)^{-1}l_3^{-1}(\gamma\beta)^{-1}yl_4) \tag{43}$$

$$\langle \mathrm{tr}(d_p)\,\mathrm{tr}(d_q)\rangle = -\frac{1}{2}\frac{1}{|\Gamma|}\sum_{\Gamma}\cdots\frac{1}{|\Gamma|}\sum_{\Gamma}\mathrm{tr}(xl_1yl_2)\ \mathrm{tr}(xl_3yl_4) \tag{44}$$

Reordering at a pair of intersections



$$\begin{aligned}\langle \mathrm{tr}(d_p)\,\mathrm{tr}(d_q)\rangle =& \frac{1}{|\Gamma|}\sum_{\Gamma}\cdots\frac{1}{|\Gamma|}\sum_{\Gamma}\mathrm{tr}(\alpha_1\beta_1 l_1\alpha_2\beta_2 l_2\gamma_1\delta_1 l_3\gamma_2\delta_2 l_4)\times \\ & \times\,\mathrm{tr}(\alpha_1\delta_1 l_7\gamma_2\beta_2 l_6\gamma_1\beta_1 l_5\alpha_2\delta_2 l_8)\end{aligned} \tag{45}$$

$$\frac{1}{|\Gamma|^2}\sum_{u\in\Gamma}\sum_{v\in\Gamma}\mathrm{tr}(uav^{-1}bu^{-1}cvd) = \frac{1}{4}\,\mathrm{tr}(cbad) \tag{46}$$

$$x_j := \alpha_j\beta_j \qquad y_j := \gamma_j\delta_j \tag{47}$$

$$\begin{aligned}\langle \mathrm{tr}(d_p)\,\mathrm{tr}(d_q)\rangle =& \frac{1}{4}\,\frac{1}{|\Gamma|}\sum_{\Gamma}\cdots\frac{1}{|\Gamma|}\sum_{\Gamma}\mathrm{tr}(x_1 l_1 x_2 l_2 y_1 l_3 y_2 l_4)\times \\ & \times\,\mathrm{tr}(x_1 l_5 x_2 l_6 y_1 l_7 y_2 l_8)\end{aligned} \tag{48}$$

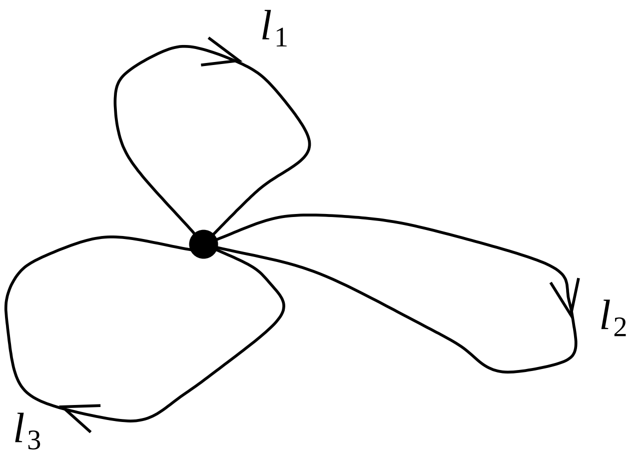
Counting self-intersections

$m-1$ self-intersections each contributing $(-1/2)$.

$$\langle \mathrm{tr}(d_p)\,\mathrm{tr}(d_q)\rangle^{m,D_n} \to \left(-\frac{1}{2}\right)^{m-1} \sum_{u\in\Gamma} \frac{1}{|\Gamma|}\big(\,\mathrm{tr}(u)\big)^2 \tag{49}$$

$$\sum_{u\in\Gamma} \frac{1}{|\Gamma|}\big(\,\mathrm{tr}(u)\big)^2 = 1 \qquad \text{(irreducible rep.)} \tag{50}$$

(Degenerate self-intersections counted according to changes in the order of q relative to p.)



Higher spins

Identities (38)–(40) depend only on the rep. being irreducible and quaternionic.

Let $\Gamma \subseteq \mathrm{SU}(2)$ and $R^s(\Gamma)$ an irrep. $\dim(R^s(\Gamma)) = 2s+1$, ie spin s.

$$\frac{1}{|\Gamma|} \sum_{u \in \Gamma} \mathrm{tr}\left(X\, R^s(u)\, Y\, R^s(u)\right) = \frac{c}{2s+1}\, \mathrm{tr}(XY^{-1}) \tag{51}$$

$$c = \begin{cases} +1 & R^s \text{ real} \\ -1 & R^s \text{ quaternionic} \end{cases}$$

But quaternionic reps. *must* have even dim, ie half-integer spin.

$$S R(u) S^{-1} = R^\star(u) \tag{52}$$

$$\begin{aligned} S^T &= S \qquad R \text{ real} \\ S^T &= -S \qquad R \text{ quaternionic} \end{aligned} \tag{53}$$

$$\det(S^T) = \det(S) \quad \text{but} \quad \det(-S) = (-1)^{\dim(R)} \det(S)$$

Summary

- Graphs quantised with Schrödinger and Dirac op.'s
- When spin-transformations generate an irred. quaternionic rep.

$$K^m_{\mathrm{sympl}}(\tau) = \left(-\frac{1}{2}\right)^{m+1} K^m_{\mathrm{orth}}(\tau) \tag{54}$$

- Results apply to subgroups $\Gamma \subseteq \mathrm{SU}(2)$ and higher spins, integer spin implies no quaternionic rep. exists.

Example: Hamilton's quaternions

$$\Gamma = \{\pm \mathrm{I}, \pm \mathrm{i}\sigma_x, \pm \mathrm{i}\sigma_y, \pm \mathrm{i}\sigma_z\} \tag{55}$$