PERIODIC ORBITS AND SPECTRAL STATISTICS IN INTERMEDIATE SYSTEMS

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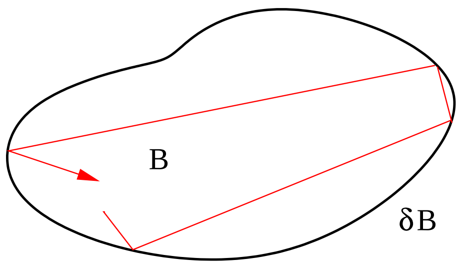
Spectral statistics

- Quantum chaos: study of classically chaotic systems
- More generally: relationship between Quantum and Classical mechanics for dynamical systems
- Main observation: Quantum systems display a level statistics which depends on their classical dynamics

What is the influence of classical dynamics on the repartition of energy levels?

Quantum systems

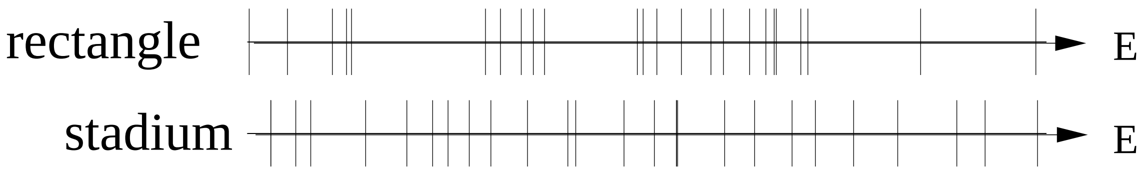


domain B
Boundary δB

We look for eigenfunctions ψ and eigenvalues E verifying $H\psi = E\psi$ with some boundary condition (e.g. Dirichlet boundary conditions $\psi|_{\delta B} = 0$).

Free motion in the billiard: the solutions of $(\Delta + E)\psi = 0$ provide us a discrete spectrum $\{E_i\}$ with density $d(E) = \sum_n \delta(E - E_n)$.

Example:

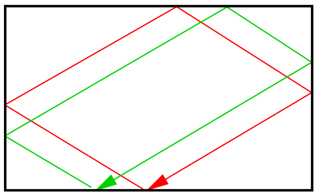


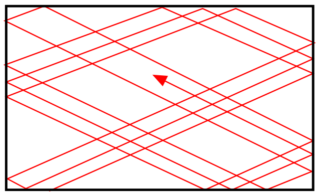
characterized by

- $P(s)$ = nearest-neighbour spacing distribution , and
- $K(\tau)$ = Fourier transform of $< d(E+\epsilon)d(E-\epsilon) >_{\mathrm{c}}$.

Integrable systems

Classical dynamics of the rectangular billiard: pencils of periodic orbits

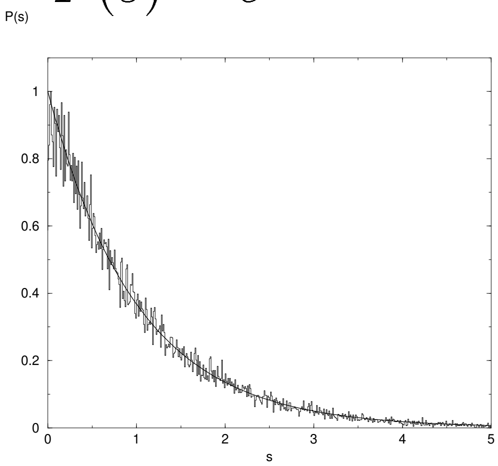




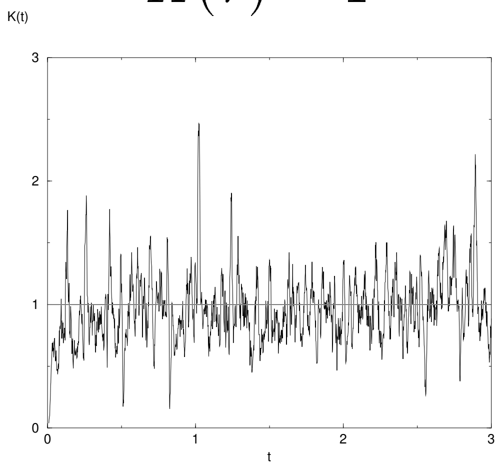
• Integrable system $\Longrightarrow$ random variables (Poisson statistics)

Berry-Tabor (1977)

$$P(s) = e^{-s}$$



$$K(\tau) = 1$$

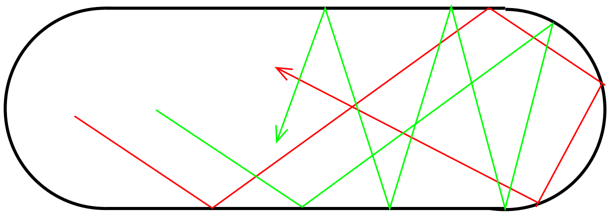


Characteristics

- no level repulsion ($P(0) = 1$)
- exponential decrease of $P(s)$
- $K(0) = 1$

Chaotic systems

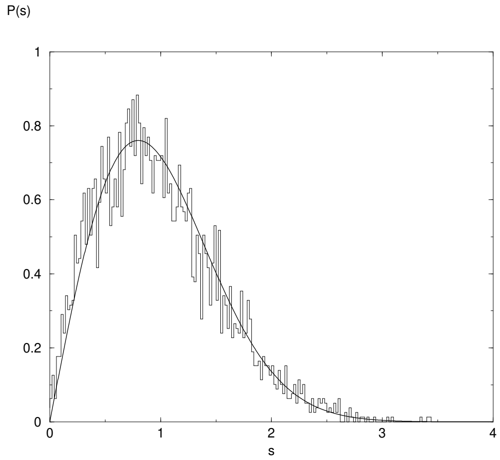
Classical dynamics of the Bunimovich billiard: divergent trajectories and isolated periodic orbits

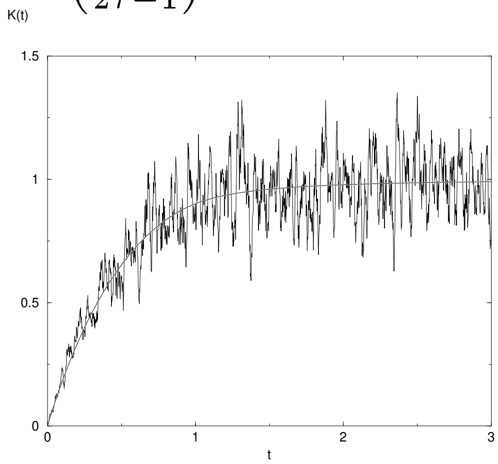


• Chaotic system $\Longrightarrow$ eigenvalues of random matrices

Bohigas-Giannoni-Schmit (1984)

$$P(s) = a_\beta s^\beta e^{-b_\beta s^2} \qquad K(\tau) = \left| \begin{array}{ll} 2\tau - \tau \ln(1+2\tau) & 0 < \tau < 1 \\ 2 - \tau \ln\left(\frac{2\tau+1}{2\tau-1}\right) & \tau > 1 \end{array} \right.$$





Characteristics

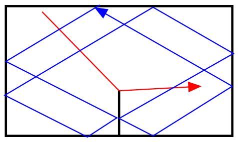
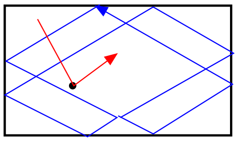
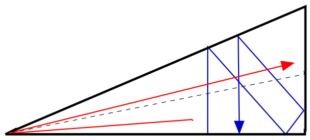
• level repulsion ($P(0) = 0$)

• behaviour at infinity $\sim e^{-as^2}$

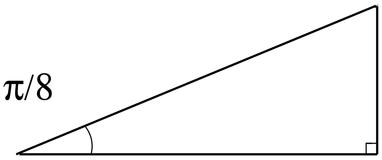
• $K(0) = 0$

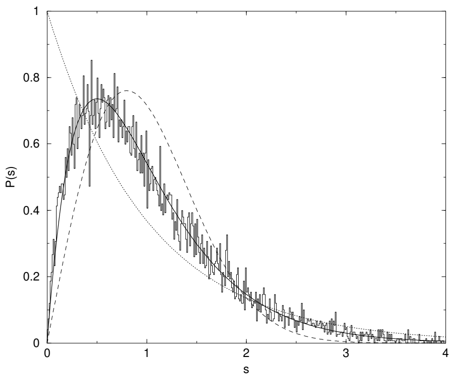
Intermediate systems

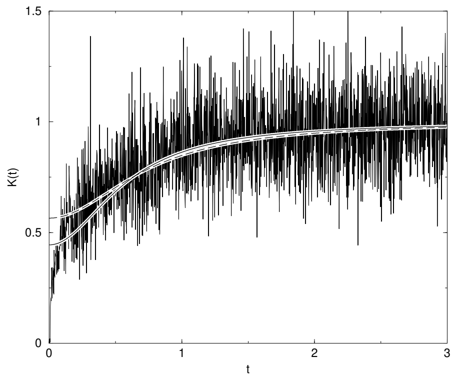
$\exists$ systems neither integrable nor chaotic: Barrier billiards, billiards with flux line, triangular billiards. . . $\Longrightarrow$ pencils of periodic orbits AND diffractive orbits for which the classical movement is not uniquely determined at the singularity.

Spectral statistics of such systems:
(Exemple:







Semi-poisson Bogomolny-Gerland-Schmit (1999) :

$$P(s) = 4se^{-2s}, \qquad K(\tau) = \frac{8 + 4\pi^2\tau^2}{16 + 4\pi^2\tau^2} \; (K(0) = 1/2)$$

Intermediate statistics

Intermediate statistics are characterized by:

- level repulsion $P(s) \underset{0}{\sim} 0$

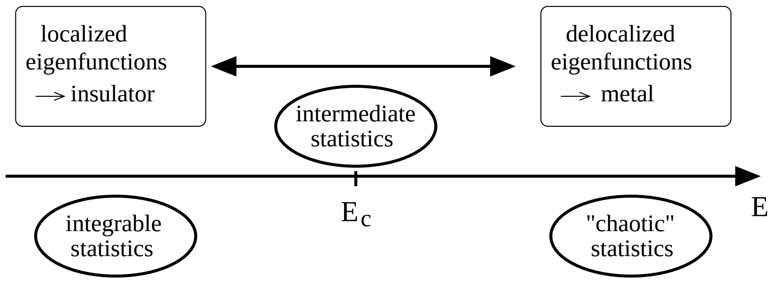
- exponential decrease $P(s) \underset{\infty}{\sim} e^{-as}$ $(a > 0)$

- form factor $0 < K(0) < 1$

- number variance $\Sigma^2(L) \underset{L \to \infty}{\sim} K(0) L$

• 3D Anderson model at metal-insulator transition:

Shklovskii *et al* (1993)



Classical dynamics of intermediate systems

• Integrable systems with point perturbation

→ same families (pencils) of periodic orbits as in the integrable system

→ existence of diffractive orbits

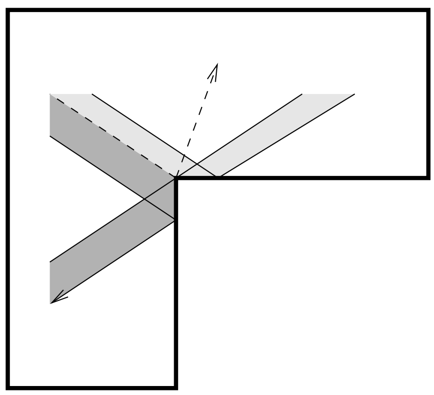
(Ex: Šeba billiard, Aharonov-Bohm billiard)

• Pseudo-integrable systems Berry-Richens (1981) :
movement in phase-space confined to a two-dimensional surface of genus $g \geq 2$

→ pencils of periodic orbits

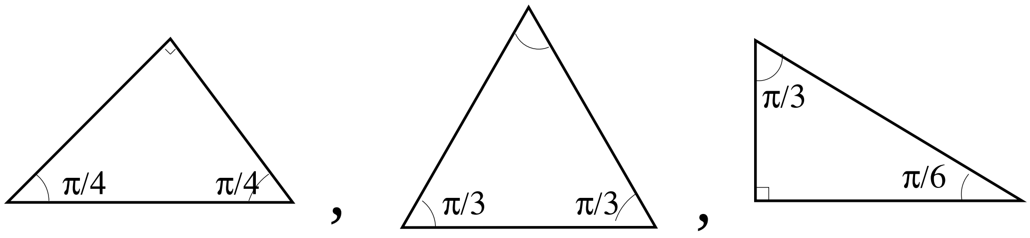
→ Chaotic feature: splitting of the pencils at vertices and existence of diffractive orbits

Ex: rational billiards
(angles $\alpha_i = \pi m_i / n_i$):

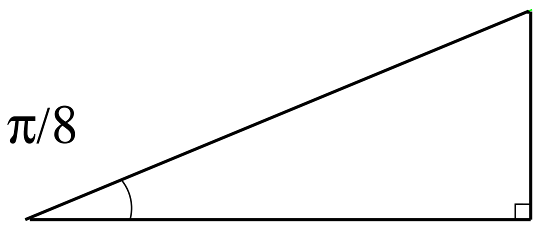


Example: triangular billiards

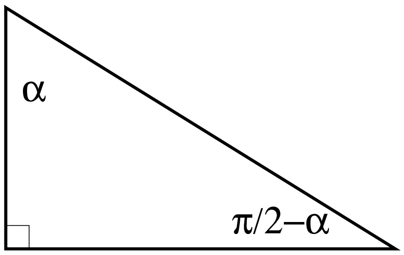
- Integrable:



- ergodic in almost all directions; non-mixing; intermediate statistics: pseudo-integrable triangles
  Artuso, guarneri, Casati 1997



- ergodic and weakly mixing: partly irrational billiards
  Artuso 1997 :



- mixing: totally irrational billiards
  Casati-Prosen 1999

Semiclassical approach

• Trace formula Balian-Bloch (1970), Gutzwiller (1971), Berry-Tabor (1976) $d(E) = \bar{d} + d_{\text{p.o.}}(E)$, with (if $k = \sqrt{E}$)

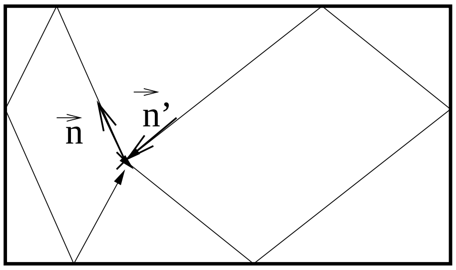
$$d_{\text{p.o.}}(E) = \sum_p \frac{\mathcal{A}_p}{2\pi} \frac{e^{ikl_p - i\pi/4 - i\nu_p \pi/2}}{\sqrt{8\pi k l_p}} + \text{c.c.}$$

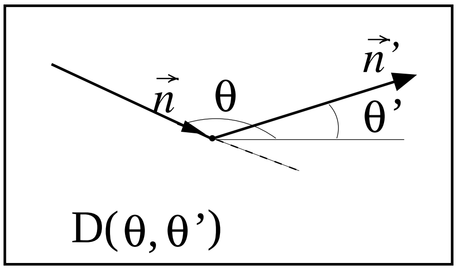
• Modified trace formula, taking into account diffractives orbits:

$$d(E) = \bar{d} + d_{\text{p.o.}}(E) + d_{\text{diff.}}(E),$$

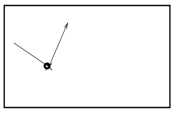
$$d_{\text{diff},m}(E) = \sum_{\vec{l}_1,\ldots,\vec{l}_m} \frac{L}{4\pi k m} \prod_{j=1}^{m} \frac{\mathcal{D}_j}{\sqrt{8\pi k l_j}} e^{ikl_j - i\nu_j \frac{\pi}{2} - 3i\frac{\pi}{4}} + \text{cc.}$$

Keller (1962), Vattay-Wirzba-Rosenqvist (1994), Sieber-Pavloff-Schmit (1997)

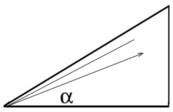




$\mathcal{D}_j$= diffraction coefficient $\mathcal{D}(\theta, \theta')$. Ex:



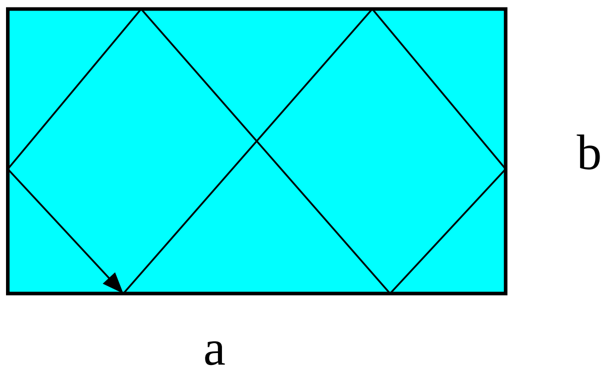
$$\mathcal{D}(\theta, \theta') = \frac{2\sin(\alpha\pi)}{\cos\frac{\theta-\theta'}{2}} \exp\left(i\frac{\theta-\theta'}{2}\right)$$



$$\mathcal{D}(\theta, \theta') = \frac{2}{N}\sin(\tfrac{\pi}{N}) \sum_{\epsilon=\pm 1} \left(\cos\frac{\pi}{N} - \epsilon\cos\frac{\theta+\epsilon\theta'}{N}\right)^{-1},$$

$$N = \frac{\alpha}{\pi}$$

The Form factor in the integrable case



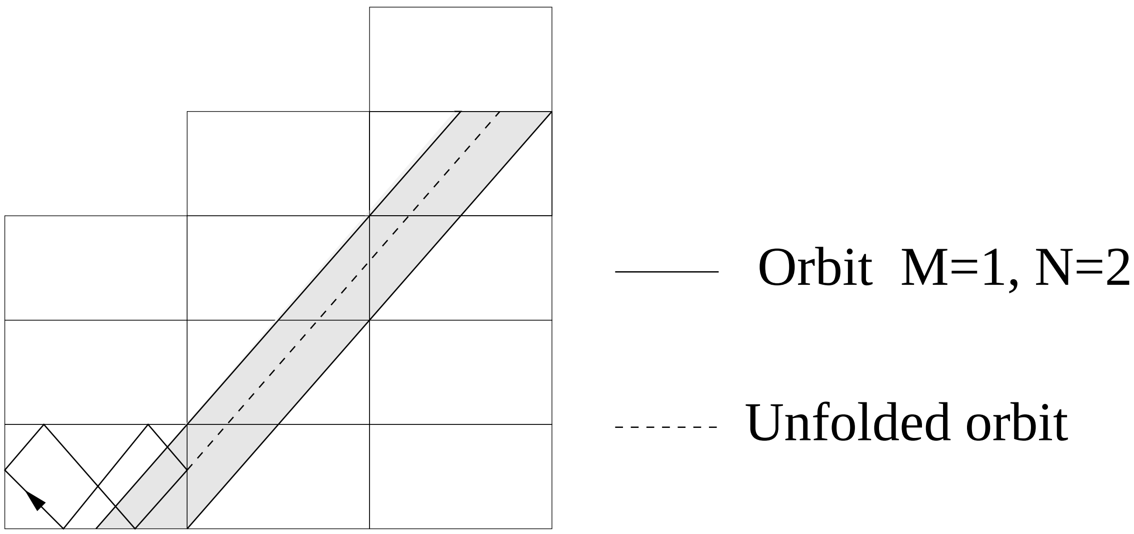
area $\mathcal{A} = a \times b$

$$K(\tau) = \frac{1}{\bar{d}} \int d\epsilon < d^{\mathrm{osc}}(E+\epsilon/2) d^{\mathrm{osc}}(E-\epsilon/2) >_E e^{2i\pi \bar{d}\tau\epsilon}$$

Trace formula $d^{\mathrm{osc}} = d_{p.o.} + d_{d.o.}$

$$d_{\mathrm{p.o.}}(E) = \sum_{M;N>0} \frac{\mathcal{A}_p}{2\pi} \frac{e^{ikl_p - i\pi/4}}{\sqrt{8\pi k l_p}} + \mathrm{c.c.}, \qquad \bar{d} = \frac{\mathcal{A}}{4\pi}$$

Periodic orbits: $l_p = \sqrt{(2Ma)^2 + (2Nb)^2}$



Semi-classical spectral form factor

$$d(E+\epsilon/2)d(E-\epsilon/2) = \frac{4\mathcal{A}^2}{\pi^2}\sum_{l_p,l_{p'}} \frac{e^{ik(l_p-lp')-i\epsilon(l_p+lp')/4k}}{\sqrt{8\pi k l_p}\sqrt{8\pi k l_{p'}}}$$

Diagonal approximation: $l_p \simeq lp'$.

$$\begin{aligned} K(\tau) &\simeq \frac{16\mathcal{A}}{\pi}\int d\epsilon \sum_{l_p} \frac{e^{-il_p\epsilon/2k}}{8\pi k l_p} e^{2i\pi\bar{d}\tau\epsilon} \\ &\simeq \frac{8\mathcal{A}}{\pi}\sum_{l_p}\frac{1}{l_p}\delta\left(l_p - 4\pi k\bar{d}\tau\right) \\ &\simeq \frac{8\mathcal{A}}{\pi}\int_0^\infty \mathrm{d}l\rho(l)\frac{1}{l}\delta\left(l-4\pi k\bar{d}\tau\right) \end{aligned}$$

Density of periodic orbits:

$$\rho(l) = \sum \delta(l-l_p) = \frac{\pi l}{8\mathcal{A}}$$

$$\rightarrow K(\tau) = 1$$

$K(0)$ in the intermediate case

Q.: What is the value of the semiclassical spectral form factor at the origin?
*$K(0) = 1$ for integrable systems
*$K(0) = 0$ for chaotic systems
* intermediate systems: $0 < K(0) < 1$?

- trace formula $d(E) = \bar{d} + d_{po}(E) + d_{do}(E)$

- contribution of $d_{\rm do}$: $\sim (L/k) \sim \tau \rightarrow_{\tau\rightarrow 0} 0$

  For diffractive systems, $d_{\rm do}$ can be neglected in the computation of $K(0)$: only periodic orbits are to be taken into account.
  But if the p.o. are not affected, $K(0) = 1$. So

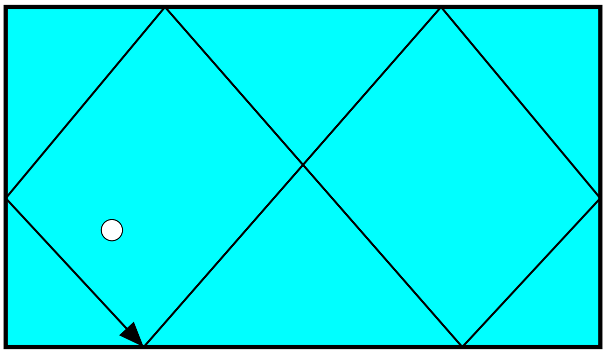
  → either the phase associated to the orbits is modified (Aharonov-Bohm billiard)
  → or the classical dynamics is different from the integrable case (e.g. on a surface of genus $g \geq 2$).

- We use

$$d(E) = \bar{d} + \sum_{po} \frac{\mathcal{A}_p}{2\pi} \frac{e^{ikl_p - i\pi/4}}{\sqrt{8\pi k l_p}} + \text{c.c.}$$

→ we need to know the lengths and areas occupied by the pencils of periodic orbits.

Aharonov-Bohm flux line

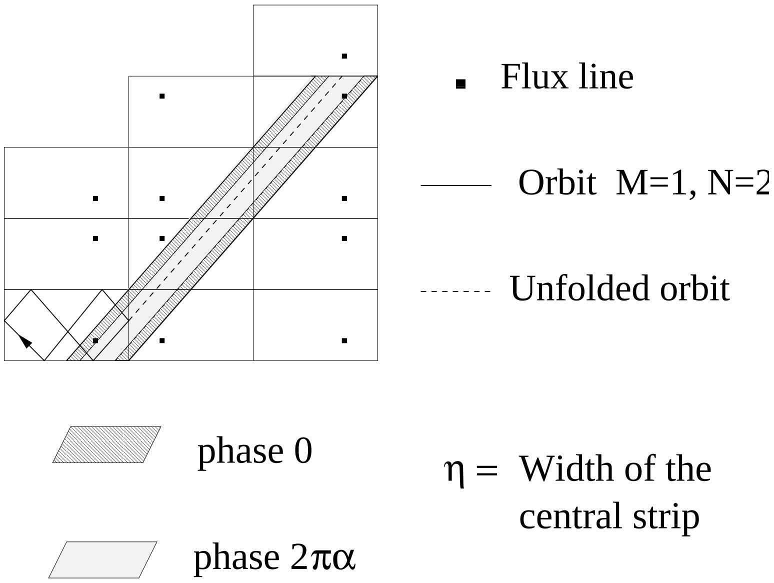


$$\vec{B} = \phi\delta(r - r_0)\vec{u}_z$$
$$\vec{r}_0 = (x_0, y_0)$$

Flux line at $\vec{r}_0$ with vector potential

$$A_r = 0, \quad A_\varphi = \frac{\alpha}{r} \qquad (\alpha = -e\phi/2\pi\hbar c)$$

An orbit acquires a phase $\pm 2\pi\alpha$ when turning around the singularity



$$\eta_p = \sum_{n=1}^{\infty} \frac{8\sin^2(\pi n M \frac{x_0}{a})\sin^2(\pi n N \frac{y_0}{b})}{\pi^2 n^2}$$

$\longrightarrow$ density of states

$$d_{\text{p.o.}}(E) = \sum_p \frac{2\mathcal{A}(1-2\eta_p \sin^2 \pi n\alpha)}{2\pi} \frac{e^{ikl_p - i\frac{\pi}{4}}}{\sqrt{8\pi k l_p}} + \text{c.c.},$$

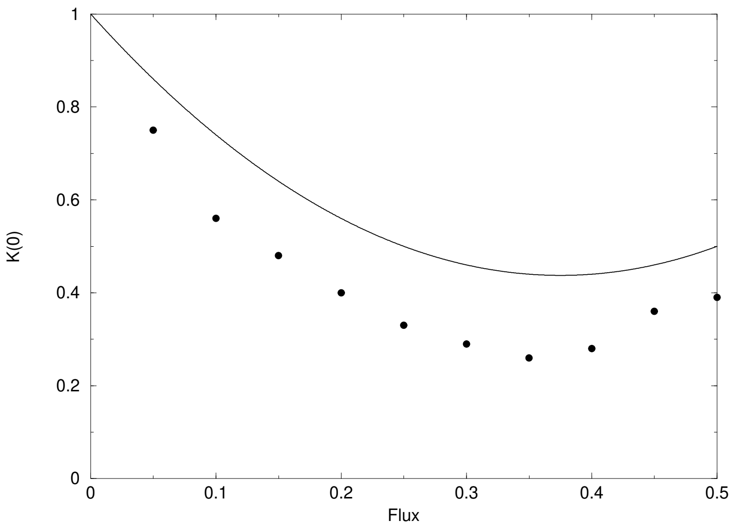
and $K(0)$:

$$K(0) = 1 - 12\bar{\alpha}(1-\bar{\alpha}) < \eta > +6\bar{\alpha} < \eta^2 >$$

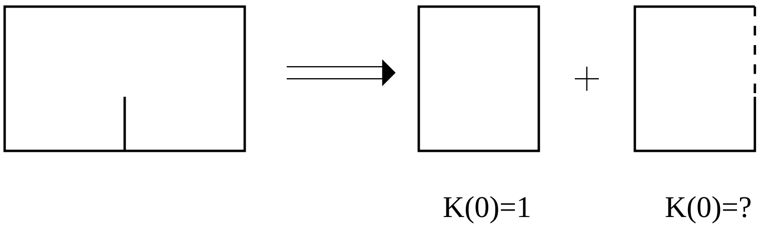
($\bar{\alpha}$=distance from α to the nearest integer: $0 \le \bar{\alpha} \le \frac{1}{2}$)

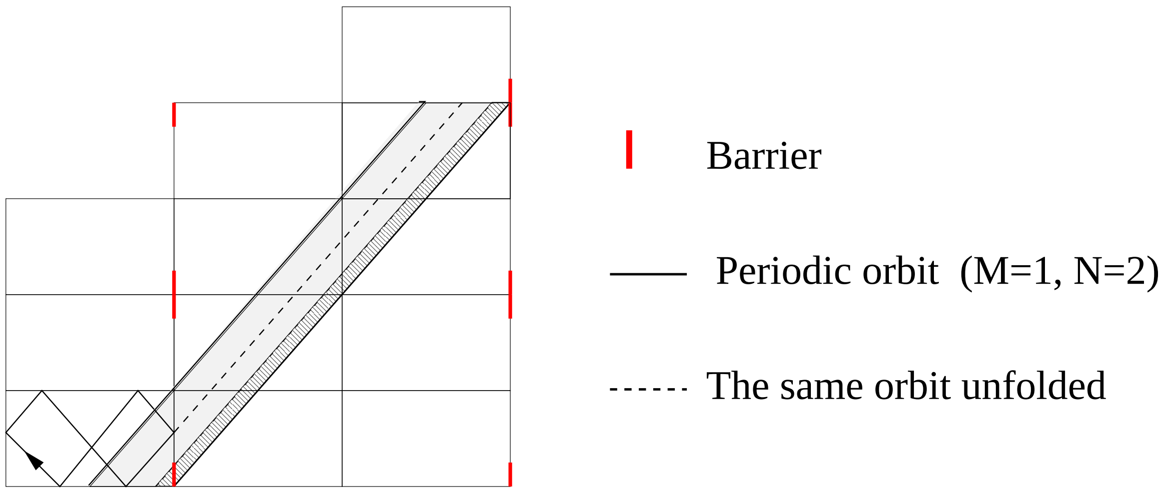
Examples:

- $x_0/a = 1/3, y_0/b = 4/7$: $K(0) = 1 - \frac{47}{252}\bar{\alpha} + 4\bar{\alpha}^2$
- $x_0/a, y_0/b$ both irrationnal: $K(0) = 1 - 3\bar{\alpha} + 4\bar{\alpha}^2$ (If $\bar{\alpha} = 1/2$, $K(0) = 1/2$)



Barrier billiards





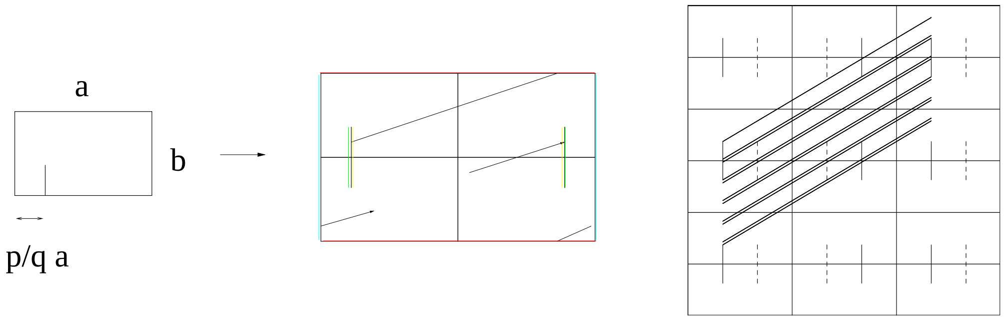
$\Longrightarrow$ $d(E)$ and $K(0)$. Here,

$$K(0) = \frac{1}{2}$$

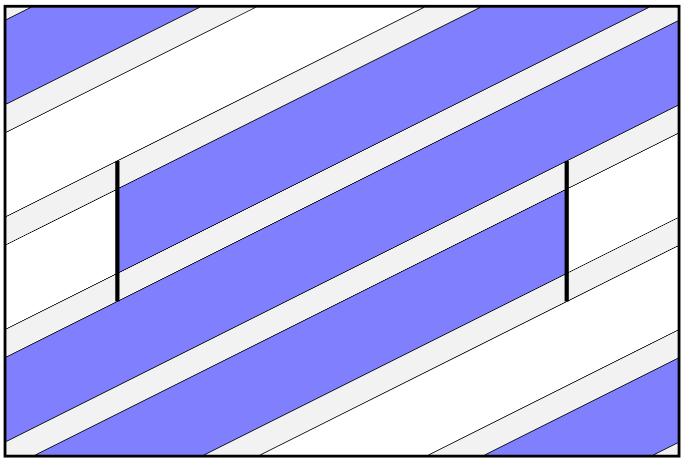
Bogomolny, Giraud 2001

Asymetric barrier billiard

- Barrier billiard



Eskin-Masur-Schmoll (2001) : there are three cylinders of periodic orbits in each direction $(M\frac{a}{q}, 2Nb)$, $M, N \in \mathbb{Z}$



with lengths $w_1 l_p, w_2 l_p, w_3 l_p$ and $l_p = \sqrt{(Ma/q)^2 + (2Nb)^2}$.

The total density of lengths is

$$\rho(l) \sim c(q)\frac{\pi l}{8\mathcal{A}}$$

with

$$c(q) = \frac{q}{N_q} \sum_{r|q} \mu(r) \sum_{\substack{(s_1,s_2)=1 \\ s_1u_1+s_2u_2=q/r}} u_1 u_2 (u_1+u_2) \min(s_1,s_2)$$

$$\times \left(\frac{1}{u_1^2} + \frac{1}{u_2^2} + \frac{1}{(u_1+u_2)^2} \right).$$

N_q =number of primitive covers of degree q of a surface of genus 2 with 2 branch points

$$N_q = \sum_{r|q} \mu(r) r^2 \sum_{\substack{(s_1,s_2)=1 \\ s_1u_1+s_2u_2=q/r}} u_1 u_2 (u_1+u_2) \min(s_1,s_2).$$

The result is

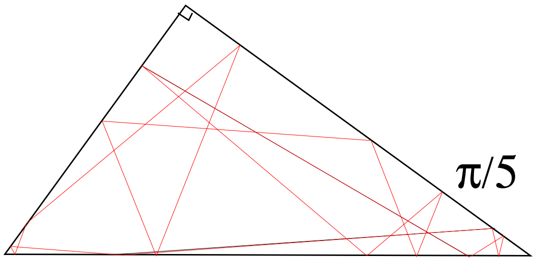
$$\begin{aligned} K(0) \quad &= \quad \frac{1}{qN_q} \sum_{\substack{(s_1,s_2)=1 \\ s_1w_1+s_2w_2=q}} \sum_{r|(w_1,w_2)} \frac{\mu(r)}{r} w_1 w_2 (w_1+w_2) \Big[\min(s_1,s_2)^3 \\ &- \quad \left(s_1 + s_2 + q \frac{\gcd(w_1,w_2)^2}{w_1 w_2 (w_1+w_2)} \right) \min(s_1,s_2)^2 \\ &+ \quad \left(s_1^2 + s_2^2 + 2 s_1 s_2 \frac{\gcd(w_1,w_2)^2}{w_1 w_2} \right) \min(s_1,s_2) \Big]. \end{aligned}$$

Final result:

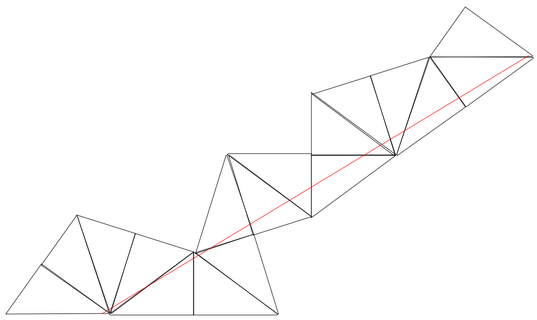
$$K(0) = \frac{1}{2} + \frac{1}{q}$$

Polygonal rational billiards

Billiards with angles $\pi m_i / n_i$
Example: $\left(\frac{\pi}{2}, \frac{\pi}{n}, \pi\frac{n-2}{2n}\right)$ -triangle



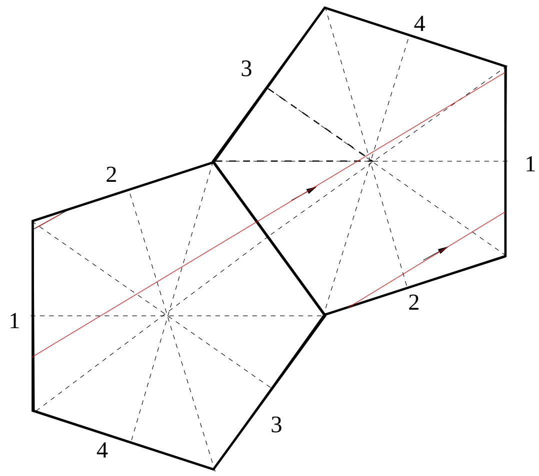
Trajectory in the triangle

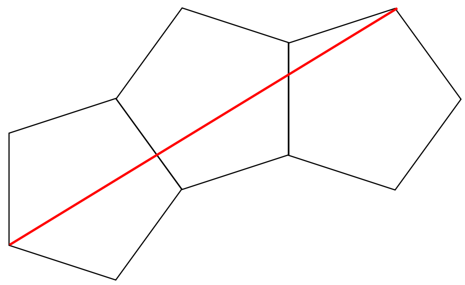


Unfolded trajectory in the plane

Equivalent structure Zemlyakov-Katok, 1976 :
$2N$ copies of the initial billard ($N =$lcm(n_i)).

$\Longrightarrow$ surface of genus $g = 1 + \frac{N}{2}\sum_i \frac{m_i - 1}{n_i}$



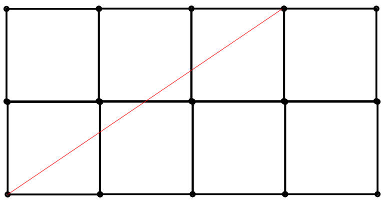


Stabilizer

• Stabilizer of a structure: discrete subgroupe $G \subset SL(2,\mathbb{R})$ such that the vertices and their images are globally invariant when G acts on $\mathbb{R}^2$.

Example: square billiard

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad SL(2,\mathbb{Z}) = < S, T >$$



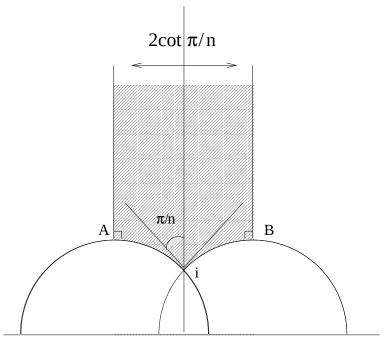
$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = T^2 S T^2 S \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In any direction, orbits are either closed or uniformly distributed.

• Here G is the group G_n generated by (n odd)

$$\sigma_n = \begin{pmatrix} \cos\frac{\pi}{n} & -\sin\frac{\pi}{n} \\ \sin\frac{\pi}{n} & \cos\frac{\pi}{n} \end{pmatrix}, \quad \tau_n = \begin{pmatrix} 1 & 2\cot\frac{\pi}{n} \\ 0 & 1 \end{pmatrix}.$$

Fundamental domain $\mathcal{D}$ of the group G_n:



union of 2 hyperbolic triangles $Ai\infty$ et $Bi\infty$, area $\pi - \hat{A} - \hat{i} - \hat{\infty}$

Veech billiards

Veech billiards: $\exists G$ discrete subgroup of $SL(2,\mathbb{R})$ such that

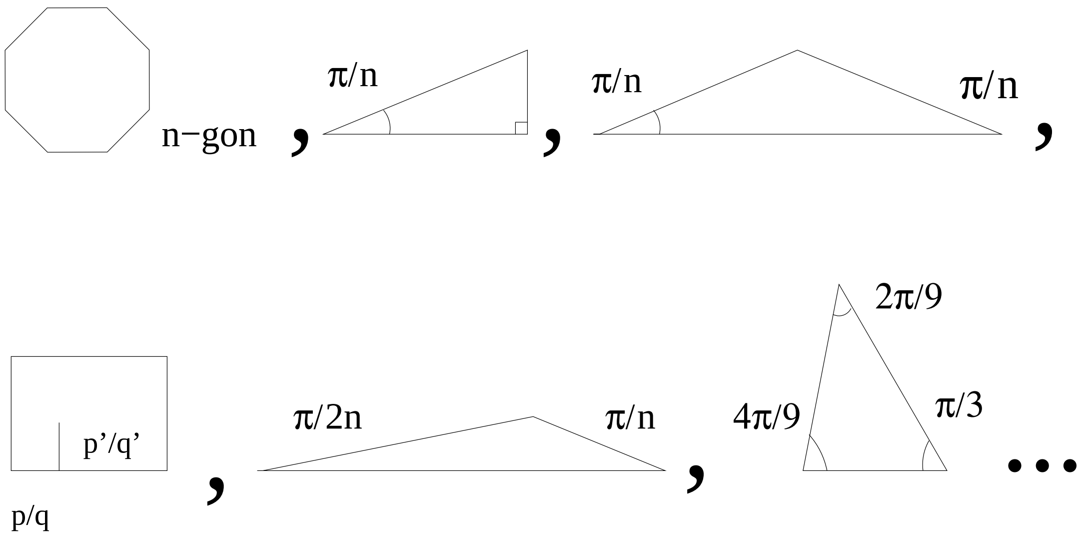
- Vol$(\mathbb{H}/G) < \infty$ $(\mathbb{H} = \{z \in \mathbb{C};\ \mathcal{I}m\ z > 0\})$
- G maps the singularities of the equivalent structure onto singularities

Then Veech (1989)

→ "Veech alternative": for any direction $v \in S^1$, EITHER all non-singular trajectories parallel to v are periodic, OR all are ergodic

→ the directions where the periodic orbits are, are the directions where the saddle-connexions are.

Examples:



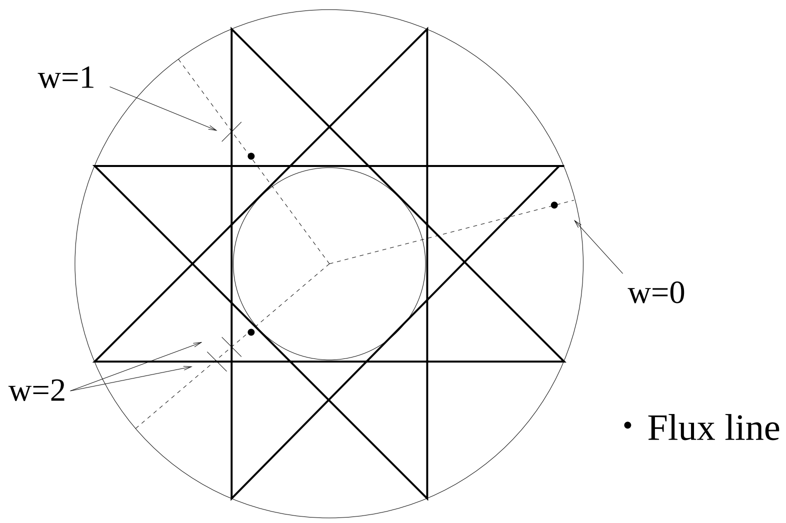
For the triangles $(\frac{\pi}{2}, \frac{\pi}{n}, \frac{\pi}{2} - \frac{\pi}{n})$,

$$K(0) = \frac{n + \epsilon(n)}{3(n-2)}, \quad \epsilon(n) = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even}, \ 3 \nmid n \\ 6 & n \text{ even}, \ 3 \mid n. \end{cases}$$

Bogomolny-Giraud-Schmit (2001)

Some other results for $K(0)$

Circular billiard with flux line



w=winding number of the orbit (M, N) around the flux line

$$K(0) = 1 - \frac{4\lambda}{\pi}\bar{\alpha}\,(1 - \bar{\alpha})$$

$\lambda = \arcsin \frac{d}{R} + \frac{d}{R}\sqrt{1 - \frac{d^2}{R^2}}$, R =radius, d =distance to the centre.

Form Factor for diffractive systems

$$K(\tau) = \frac{1}{\bar{d}} \int d\epsilon < d^{\text{osc}}(E{+}\epsilon/2) d^{\text{osc}}(E{-}\epsilon/2) >_E e^{2i\pi \bar{d} \tau \epsilon}$$

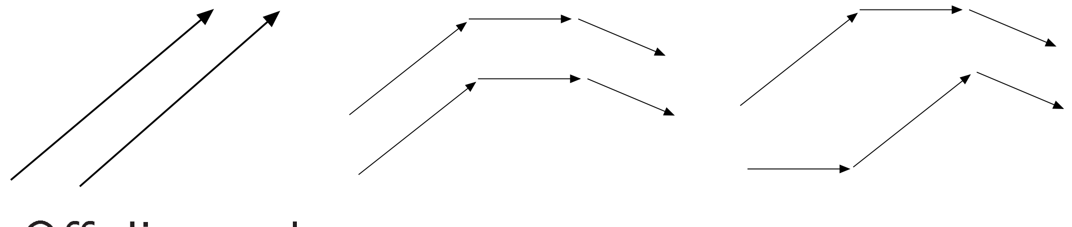
We want a development into powers of (L/k) of

$$< d^{\text{osc}}(E + \epsilon/2) d^{\text{osc}}(E - \epsilon/2) >$$

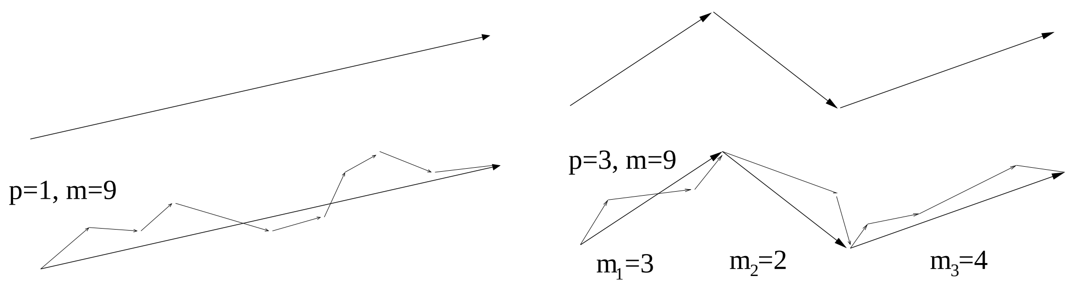
- $d^{\text{osc}}(E) = \sum_{\vec{l}_p} \mathcal{A}_p e^{ikl_p} + \sum_m \sum_{\vec{l}_{d1},\ldots,\vec{l}_{dm}} \mathcal{A}_d e^{ik(l_{d1}+\cdots+l_{dm})}$

- Interaction between periodic and diffractive orbits:

  $*$ Diagonal terms



  $*$ Off-diagonal terms



Computation of $K(\tau)$

• If $\mathcal{D}$ (diffraction coefficient) is constant: all-order expansion of $K(\tau)$, taking into account the interactions between diagonal and non-diagonal terms and both periodic and diffractive orbits:

$$K(\tau) \;= < e^{-\frac{|\mathcal{D}|^2 g\tau}{16} r} >$$

$$+\sum_{p=2}^{\infty}\frac{1}{p!}\sum_{m,n\geq 0} A_{mnp}\left(\frac{-i\mathcal{D}g}{8}\right)^{m+p}\left(\frac{i\bar{\mathcal{D}}g}{8}\right)^{n+p}\tau^{m+n+p+1},$$

$$A_{mnp} = \frac{(m+p-1)!(n+p-1)!}{(m+n+p-1)!}\sum_{\substack{m_i\geq 0\\ n_j\geq 0}}\prod_{i=1}^{p}\left[\frac{C_{m_i+n_i}^{m_i} < r^{m_i+n_i+2} >}{(m_i+1)!(n_i+1)!}\right]$$

$$\begin{array}{ll}
\text{Periodic:} & r = 1 \;\; \rightarrow < r^n >= 1 \\
\text{Dirichlet:} & r = 4\sin^2\phi_1 \sin^2\phi_2 \\
\text{with} & \left|\begin{array}{ll} \phi_1, \phi_2 \in [0,\pi] & (x_0, y_0 \in \mathbb{R}\setminus\mathbb{Q}) \\ \phi_i = \pi\frac{r_i}{q_i} & x_0 = \frac{p_1}{q_1}, y_0 = \frac{p_2}{q_2}) \end{array}\right.
\end{array}$$

$$\rightarrow \text{ e.g. } < r^n >= \left(\frac{(2n)!}{2^n (n!)^2}\right)^2 \text{ in the irrational case}$$

• First terms $K(\tau) = 1 - a\tau + b\tau^2 + o(\tau^2),\; a, b > 0$
$\rightarrow$ different from integrable or chaotic systems
• corresponds to the exact computation (FT of $R_2(s)$)
• Diffractive coefficient $\mathcal{D}$ depending on the angles: explicit sum over all orbits

Conclusion

$\rightarrow$ There exist systems for which the form factor verifies $0 < K(0) < 1$

$\rightarrow$ For a diffractive system with finite diffraction coefficient the periodic and diffractive orbits interact and the diagonal and off-diagonal terms give the expansion of $K(\tau)$ at all order.

$\rightarrow$ It has been proved that there exists a system, the Šeba billiard, which verifies $P(s) \sim_0 0$ and $P(s) \sim_\infty e^{-as}, a > 0$ Bogomolny-Gerland-Schmit, 2001

Bogomolny-Giraud, 2001