SS 2001 Abt. Theoretische Physik
Problem sheet 2 J. Bolte and S. Keppeler

Theory of Quantum Chaos

4. **Circle Billiard (classical)**

   A pointlike particle with mass m moves freely inside the unit disc $D := \{x = (x_1, x_2); |x| \leq 1\}$ with elastic reflections at the boundary $\partial D = \{x; |x| = 1\}$.

   (a) Determine the Lagrangian in polar coordinates $(x_1, x_2) = r(\cos\varphi, \sin\varphi)$ and the canonical momenta conjugate to (r, φ). Work out also the Hamiltonian and Hamilton's equations of motion for the system.

   (b) Are there any conserved quantities besides the energy? (Hint: Don't ignore the reflection!)

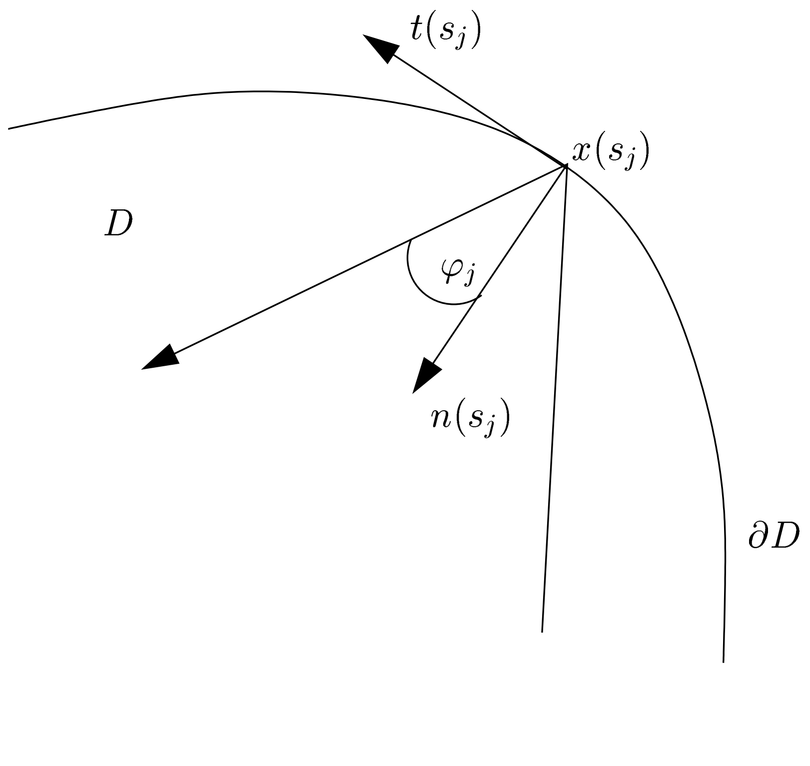
   (c) Express angular momentum in polar coordinates. What is the modulus of the angular momentum in terms of the energy and the angle of reflection (sketch!)?

   (d) We characterise the phase space of the circle billiard by considering the set M of points which share the same values of *all* conserved quantities. (Hint: Express p_r in terms of the conserved quantities and the variable r.) Sketch M in an (x_1, x_2, p_r)-plot.

   (e) Determine the billiard map in polar coordinates and in Birkhoff coordinates.

   (f) For which initial conditions does one obtain periodic orbits? How can they be characterised?

   (g) Sketch trajectories (periodic and non-periodic) in the circle billiard (in x-space). How are they related to the projection of M onto the x_1x_2-plane.



5. **Kicked Rotator / Standard Map (ii)**

A function $R(q', q)$ is called generating function of the map

$$F : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p' \\ q' \end{pmatrix}$$

if

$$p = -\frac{\partial R}{\partial q} \quad \text{and} \quad p' = \frac{\partial R}{\partial q'} .$$

Compare with generating functions of canonical transformations of type F_1 in Goldstein's notation.

(a) Determine a generating function of the standard map (cf., exercise 3, $m = 1$).

(b) Calculate the Lagrangian $L(q, \dot{q})$ of the kicked rotator and the action

$$\lim_{\epsilon \to 0+} \int_{n+\epsilon}^{n+1+\epsilon} L(q, \dot{q}) \, \mathrm{d}t$$

corresponding to one step of the standard map. Compare with (a).

(c) Make a linear stability analysis of all fixed and periodic points of the standard map which where determined in exercise 3, i.e. consider the linearised map

$$F \begin{pmatrix} p \\ q \end{pmatrix} = F \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} + F' \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \begin{pmatrix} p - p_0 \\ q - q_0 \end{pmatrix} + \mathcal{O}\left((p - p_0)^2 + (q - q_0)^2\right)$$

and calculate the eigenvalues of F'. What is the meaning of the eigenvectors of $F' \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right)$?

(d) Illustrate the results of (c) with the help of our computer programme, i.e. iterate initial points close to a fixed/periodic point for *interesting* values of the parameter k.

6. **Ergodicity**

In analogy to a condition given in the lecture we say that a map

$$\begin{aligned} T : M &\to M \\ x &\mapsto T(x) \end{aligned}$$

is ergodic if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f\left(T^n(x_0)\right) = \frac{1}{|M|} \int_M f(x) \, \mathrm{d}x \qquad (*)$$

$\forall \, f \in L^2(M)$ and for almost all $x_0 \in M$.

Remark: For a more general definition and consequences see, e.g., R. Mañé *Ergodic Theory and Differentiable Dynamics* Springer 1987.

(a) Prove: $(*)$ is equivalent to
$(+)$ $f \in L^2(M)$ is invariant under T (i.e. $f \circ T = f$)
$\Leftrightarrow$ f is constant almost everywhere.

(b) Show that rotations of the circle

$$\begin{aligned} T_\alpha : S^1 &\to S^1 \\ x &\mapsto x + \alpha \qquad \mod 1 \end{aligned}$$

are ergodic if and only if α is irrational.
Hint: Use property $(+)$ and expand $f(x)$ in a Fourier series.