Abt. Theoretische Physik
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Theory of Quantum Chaos

7. **Circle Billiard (quantum mechanical)**
The quantum mechanical circle billiard is described by the Hamiltonian $\hat{H} = -\frac{\hbar^2}{2m}\Delta$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ denotes the Laplacian in two dimensions. For the eigenfunctions we require Dirichlet boundary conditions on ∂D, i.e.

$$\begin{aligned} \hat{H}\psi(x) &= E\psi(x)\,, \\ \psi(x) &= 0 \quad \text{for} \quad |x| = 1\,. \end{aligned}$$

(a) Express Δ in polar coordinates and separate variables in the eigenvalue problem by making the ansatz $\psi(x) = f(\varphi)u(r)$. Which differential equations do you obtain for $f(\varphi)$ and $u(r)$? Solve the equation for $f(\varphi)$.

(b) Express $u(r)$ in terms of Bessel functions $J_l(z)$ which solve the differential equation

$$z^2 J_l''(z) + zJ_l'(z) + (z^2 - l^2)J_l(z) = 0\,. \tag{$*$}$$

Since $(*)$ is a linear differential equation of second order there is second linearly independent solution besides $J_l(z)$. Why is this solution not relevant for our problem?

(c) The energy eigenvalues are obtained from the boundary condition $u(1) = 0$. Denote by $j_n(l)$, $n \in \mathbb{N}$, the n^{th} zero of $J_l(z)$ and express the eigen-energies in terms of $j_n(l)$.

(d) For fixed l and $z \to \infty$ the Bessel functions are asymptotically given by

$$J_l(z) \sim \sqrt{\frac{2}{\pi z}} \left(\cos\left(z - l\frac{\pi}{2} - \frac{\pi}{4} \right) + \mathcal{O}(z^{-1}) \right)\,.$$

Use this relation in order to determine an approximation for the eigen-energies of the quantum circle billiard in a suitable limit.

8. **Poisson summation formula**
Prove the Poisson summation formula in d dimensions

$$\sum_{n\in\mathbb{Z}^d} f(n) = \sum_{\nu\in\mathbb{Z}^d} \hat{f}(\nu)\,,$$

where $\hat{f}$ denotes the Fourier transform

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)\,\mathrm{e}^{-2\pi\mathrm{i}x\xi}\,\mathrm{d}x$$

of f. What conditions on f and $\hat{f}$ does one need for the sums to converge?
Hint: Consider the function $F(x) := \sum_{n\in\mathbb{Z}^d} f(x+n)$ and expand it in a Fourier series.

9. **Gutzwiller trace formula for the harmonic oscillator**
The quantum harmonic oscillator is defined by the Hamiltonian $\hat{H} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{m}{2}\omega^2 x^2$. Its classical analogue was treated in exercise 1. The eigenvalues of $\hat{H}$ are known to be $E_n = \hbar\omega(n+\frac{1}{2})$, $n\in\mathbb{N}_0$. The spectral density, or density of states, is defined by

$$d(E) := \sum_{n\in\mathbb{N}_0} \delta(E-E_n)\,. \tag{1}$$

By (formally) applying the Poisson summation formula to the n-sum (note that it runs over $\mathbb{N}_0$ rather than over $\mathbb{Z}$) show that the spectral density can be written in the form

$$d(E) = \frac{|\Omega_E|}{2\pi\hbar} + \sum_{k=1}^{\infty} \mathcal{A}_k \cos\left(\frac{1}{\hbar}kS\right)\,. \tag{2}$$

Here $\Omega_E := \{(p,q)|\, H(p,q) = E\}$ denotes the energy shell of the classical harmonic oscillator (cf. exercise 1) with volume

$$|\Omega_E| = \int_{\mathbb{R}^2} \delta(H(p,q)-E)\,\mathrm{d}p\mathrm{d}q\,,$$

and S denotes the action $\oint p\,\mathrm{d}q$ which was determined in exercise 1. Express the amplitudes $\mathcal{A}_k$ in terms of the period $T = \mathrm{d}S/\mathrm{d}E$ which was also calculated in exercise 1.
Illustration: Evaluate $d(E)$ on a normalised Gaussian

$$\varrho(E,E') := \frac{1}{\sqrt{2\pi\varepsilon^2}}\exp\left(-\frac{(E-E')^2}{2\varepsilon^2}\right), \quad \varepsilon > 0\,,$$

i.e. consider $\int d(E')\,\varrho(E,E')\,\mathrm{d}E'$. Show that this leads to a Gaussian damping of the contribution of long periodic orbits in the trace formula. Evaluate the periodic orbit sum for $\int d(E')\,\varrho(E,E')\,\mathrm{d}E'$ numerically with some ε using a reasonable cut-off in the k-sum.
Hint: This can, e.g., be done using Maple.