2004

ANALYTICAL MECHANICS
Exercise: Wigner-Weyl calculus

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1. (a) For $\hat{p}$ and $\hat{x}$, the momentum and position operators (with $[\hat{p},\hat{x}] = \frac{\hbar}{\mathrm{i}}$), show

$$\hat{x}\,\mathrm{e}^{-\lambda\hat{p}} = \mathrm{e}^{-\lambda\hat{p}}(\hat{x} + \lambda[\hat{p},\hat{x}])\,.$$

   Hint: Show that both sides of the equation solve the same initial value problem in λ.

   *(b) For two operators $\hat{A}$ and $\hat{B}$ define

$$\mathrm{ad}_{\hat{A}}\,\hat{B} := [\hat{A},\hat{B}]\,.$$

   Show:

$$\begin{aligned}\mathrm{e}^{\lambda\hat{A}}\hat{B}\mathrm{e}^{-\lambda\hat{A}} &= \mathrm{e}^{\lambda\mathrm{ad}_{\hat{A}}}\,\hat{B}\\ &= \hat{B} + \lambda[\hat{A},\hat{B}] + \tfrac{\lambda^2}{2}[\hat{A}[\hat{A},\hat{B}]] + \mathcal{O}(\lambda^3)\,.\end{aligned}$$

   Hint: Show that both sides of the equation solve the same initial value problem in λ.
   Remark: For $\hat{A} = \hat{p}$ and $\hat{B} = \hat{x}$ this implies (a).

   (c) Derive the following special case of the Baker-Campbell-Hausdorff rule

$$\mathrm{e}^{-(a\hat{x}+b\hat{p})} = \mathrm{e}^{-a\hat{x}}\,\mathrm{e}^{-b\hat{p}}\,\mathrm{e}^{\frac{ab}{2}[\hat{p},\hat{x}]}\,.$$

   Hint: Show that the functions $f(t) := \mathrm{e}^{-t(a\hat{x}+b\hat{p})}$ and $g(t) := \mathrm{e}^{-ta\hat{x}}\,\mathrm{e}^{-tb\hat{p}}\,\mathrm{e}^{t^2\frac{ab}{2}[\hat{p},\hat{x}]}$ solve the same initial value problem in t (use (a) along the way).

2. Calculate the operators with Weyl symbol

   (a) $A = x^2 p$.
   Express the result in terms of $\hat{p}$ and $\hat{x}$ in two ways:
   (i) With all operators $\hat{x}$ moved to the left.
   (ii) As sums of products of $\hat{p}$ and $\hat{x}$ without using additional additive constants.

   (b) $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}$, where $\boldsymbol{x}, \boldsymbol{p} \in \mathbb{R}^3$.
   Also calculate the Poisson brackets $\{L_j, L_k\}$ and the commutators $[\hat{L}_j, \hat{L}_k]$.

3. Calculate the Weyl symbol of the operator

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta + V(x)$$

   with $x \in \mathbb{R}^d$ and $\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$.

4. Calculate the marginals

$$\int_{\mathbb{R}^d} W[\psi](p,x)\,\mathrm{d}^d p \quad \text{and} \quad \int_{\mathbb{R}^d} W[\psi](p,x)\,\mathrm{d}^d x$$

of the Wigner function

$$W[\psi](p,x) = \int_{\mathbb{R}^d} \psi(x+\tfrac{z}{2})\,\overline{\psi(x-\tfrac{z}{2})}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}pz}\,\mathrm{d}^d z\,.$$

and express them in terms of the wave function $\psi(x)$ and its (normalised) Fourier transform $g(p) = (2\pi\hbar)^{-d/2}\,\tilde{\psi}(p)$, respectively.

5. Calculate the Wigner function of the coherent state

$$\psi(x) = \left(\frac{1}{\pi}\right)^{1/4} \mathrm{e}^{-\frac{1}{2}(x-\alpha)^2}$$

with $\alpha \in \mathbb{C}$. What is the meaning of $\mathrm{Re}\alpha$ and $\mathrm{Im}\alpha$?

6. (a) Derive the equation of motion for the Wigner function $W[\psi](p,x)$ in the Schrödinger picture.
   HINT: First derive the equation of motion for $\hat{P}_\psi$ using the Schrödinger equation. Then calculate the symbol of both sides and use a relation from the lecture.
   (b) Write the solution of (a) more explicitly in the case where $H(p,x) = \frac{p^2}{2m} + V(x)$.
   (c) Which equation do you obtain from the result of (a) in leading order as $\hbar \to 0$?