Eberhard Karls Universität Tübingen Sommersemester 2019
Fachbereich Mathematik
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Algebraische Kurven

Übungsaufgaben zum 1. Tutorium am 25.04.2019

Aufgabe 1 (Keine Abgabe, Präsenzübung).
Let $I, J \subset \mathbb{C}[X,Y]$ be ideals, and let $S, T \subseteq \mathbb{C}^2$ be subsets. Let $C \subset \mathbb{C}^2$ be an affine curve, given by a non-constant polynomial $f_0 \in \mathbb{C}[X,Y]$. Let $J_0 :=< f_0 >$ be the ideal generated by f_0. Prove the following implications and equalities:

$$\begin{array}{ccc} I \subseteq J & \Rightarrow & V(J) \subseteq V(I) \\ S \subseteq T & \Rightarrow & I(T) \subseteq I(S) \\ V(I \cap J) & = & V(I) \cup V(J) \\ V(I + J) & = & V(I) \cap V(J) \\ V(I(C)) & = & C \\ I(V(J_0)) & = & \sqrt{J_0} \end{array}$$

where $\sqrt{J_0} := \{g \in \mathbb{C}[X,Y]$ such that $g^n \in J_0$ for some $n \in \mathbb{N}\}$ is the *radical ideal* of J_0.

Aufgabe 2. (Keine Abgabe, Präsenzübung)
a) Consider points $P, P', Q, Q' \in \mathbb{P}^2$ with $P \neq P'$ and $Q \neq Q'$. Show that there exists a projective transformation $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\varphi(P) = P'$ and $\varphi(Q) = Q'$

b) Let $C \subset \mathbb{P}^2$ be a projective plane curve, given by a homogeneous polynomial $F \in \mathbb{C}[X,Y,Z]$. Let $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ be a projective transformation. Show that $\varphi^{-1}(C)$ is again a projective plane curve, which is given by $F \circ \varphi$.

Aufgabe 3. (Keine Abgabe, Präsenzübung)
Show that for the *real* affine plane curve $C = V(X^4 - X^2 + Y^2) \subset \mathbb{R}^2$ there exist two *double tangents*, i.e. lines, which are tangent to C in two different points each. Start by drawing a picture of C.

Aufgabe 4.
For the given projective plane curves $C \subset \mathbb{P}^2$ determine for all points $P \in C$ the multiplicity $m_P(C)$. For all singular points, find the geometric tangents.

a) $C := V(Y^2 Z - X^3)$
b) $C := V(Y^2 X^2 - Y^3 Z - Y^2 Z^2)$.

Aufgabe 5.
Let $C_1, C_2 \subset \mathbb{C}^2$ be two plane affine curves without common components. Show that $i_P(C_1, C_2) \geq 2$ if and only if C_1 and C_2 have a common tangent in P. (You may assume $P = (0,0)$, P is smooth on both C_1 and C_2, and $L_Y = V(Y)$ is a tangent to C_2 in P.)

Aufgabe 6. (Keine Abgabe, Präsenzübung)
Let $f \in \mathbb{R}[X,Y]$ be a non-constant polynomial, and let $C = V(f) \subset \mathbb{R}^2$ be a *real* affine plane curve. Let $P \in C$ be an isolated point of C, i.e. there exists an $\varepsilon > 0$ such that $U_\varepsilon(P) \cap C = \{P\}$. Show that $f : \mathbb{R}^2 \to \mathbb{R}$ has a local extremum at P, and deduce that P is a singular point of C.

Abgabe der Lösungen zu Aufgaben 4 und 5 am 25.04.2018 in der Vorlesung.