6 Tensors products

Compared to the elegant simplicity of linear maps, the handling of multilinear maps feels somewhat clumsy, compare for example remark 5.14 **??**. The basic idea of multilinear algebra is to achieve a sort of "linearization": to give a one-to-one translation of multilinear maps into linear maps. For these linear maps then the full machinery of linear algebra will be available, like representations by matrices, kernels and cokernels, and so on.

In the case of bilinear maps originating from a pair of modules $M \times N$, the price for obtaining such a linearization is that we need to understand a single "universal" bilinear map $\tau : M \times N \to T$. It has the property, that any other *bilinear* map can be uniquely derived from τ by composition with a *linear* map. This is what we will call a tensor product in definition 6.2 below.

Throughout this section, let $(R, +, \cdot)$ denote a commutative ring with a multiplicative identity element.

6.1 Example. Let $(K, +, \cdot)$ be a field, and let $m, n \in \mathbb{N}_{>0}$. Consider the K-vector spaces $U := K^n$ and $V := K^m$, together with the K-vector space of matrices $T := \mathrm{Mat}(n, m, K)$. By the rules of matrix multiplication, the map

$$\begin{array}{rccc} \tau : & U \times V & \to & T \\ & (u, v) & \mapsto & {}^t u \cdot v \end{array}$$

is clearly bilinear.

a) In general, the map τ is not surjective. For example, for $n = m = 2$, the unit matrix is not contained in the set-theoretic image of τ.

b) For the image of τ holds $\mathrm{im}(\tau) = T$. Indeed, let $\{e_1, \ldots, e_n\} \subseteq U$ and $\{f_1, \ldots, f_m\} \subseteq V$ denote the standard bases. Then the set of matrices $E_{i,j} := \tau(e_i, f_j)$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ forms the standard basis of T. In particular, we have

$$\mathrm{im}(\tau) \supseteq \mathrm{span}_K\{E_{i,j} : 1 \leq i \leq n,\ 1 \leq j \leq m\} = T,$$

so the image of the bilinear map τ equals T.

c) Let Z be an arbitrary K-vector space, and let $\varphi : U \times V \to Z$ be a bilinear map. On the standard basis of T, we define a K-linear map by

$$\begin{array}{rccc} \tilde{\varphi} : & T & \to & Z \\ & E_{i,j} & \mapsto & \varphi(e_i, f_j) \end{array}$$

Then the identity $\tilde{\varphi} \circ \tau = \varphi$ holds. Indeed, let $u = (u_1, \ldots, u_n) \in U$ and $v = (v_1, \ldots, v_m) \in V$ be arbitrary vectors. Then we compute

$$\begin{array}{rcll} \tilde{\varphi} \circ \tau(u,v) & = & \tilde{\varphi}\left(\tau\left(\sum_{i=1}^n u_i e_i, \sum_{j=1}^m v_j f_j\right)\right) & \\ & = & \tilde{\varphi}\left(\sum_{i=1}^n \sum_{j=1}^m u_i v_j \, \tau(e_i, f_j)\right) & (\tau \text{ is bilinear}) \\ & = & \tilde{\varphi}\left(\sum_{i=1}^n \sum_{j=1}^m u_i v_j \, E_{i,j}\right) & (\text{definition of } E_{i,j}) \\ & = & \sum_{i=1}^n \sum_{j=1}^m u_i v_j \, \tilde{\varphi}(E_{i,j}) & (\tilde{\varphi} \text{ is linear}) \\ & = & \sum_{i=1}^n \sum_{j=1}^m u_i v_j \, \varphi(e_i, f_j) & (\text{definition of } \tilde{\varphi}) \\ & = & \varphi\left(\sum_{i=1}^n u_i e_i, \sum_{j=1}^m v_j f_j\right) & (\varphi \text{ is bilinear}) \\ & = & \varphi(u,v). & \end{array}$$

d) Since we defined $\tilde{\varphi}$ on the standard basis of T, it is necessarily the unique K-linear map that satisfies $\tilde{\varphi} \circ \tau = \varphi$.

6.2 Definition. Let M and N be R-modules. A *tensor product of M and N over R* is a bilinear map $\tau : M \times N \to T$ into an R-module T, which satisfies the following *universal property of the tensor product*:
For any R-module Z, and any bilinear map $\varphi : M \times N \to Z$, there exists a unique homomorphism of R-modules $\tilde{\varphi} : T \to Z$, such that the diagram

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[d, "\tau"'] & Z \\ T \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

commutes.

6.3 Remark. The defining universal property characterizes a tensor product uniquely up to a unique isomorphism: if $\tau : M \times N \to T$ and $\tau' : M \times N \to T'$ are both tensor products of M and N over R, then there exists a unique isomorphism of R-modules $\alpha : T \to T'$ such that $\tau' = \alpha \circ \tau$.

6.4 Example. The bilinear map $\tau : K^n \times K^m \to \mathrm{Mat}(n, m, K)$ from example 6.1 is a tensor product over K.

6.5 Proposition. *A bilinear map $\tau : M \times N \to T$ is a tensor product of M and N over R, if and only if for all R-modules Z the map*

$$\begin{array}{rccc} \Lambda : & \mathrm{Hom}_R(T, Z) & \to & L_R(M, N; Z) \\ & \tilde{\varphi} & \mapsto & \tilde{\varphi} \circ \tau \end{array}$$

is an isomorphism on R-modules.

Proof. Note that by lemma **??** the composition $\tilde{\varphi} \circ \tau : M \times N \to Z$ is indeed a bilinear map. One verifies immediately that Λ is a homomorphism of R-modules.

Suppose that τ is a tensor product. Then the universal property of τ implies that for any bilinear map $\varphi \in L_R(M, N; Z)$, there exists a R-linear map $\tilde{\varphi} \in \mathrm{Hom}_R(T, Z)$, such that $\varphi = \tilde{\varphi} \circ \tau = \Lambda(\tilde{\varphi})$, and this $\tilde{\varphi}$ is unique. Hence Λ is surjective and injective.

Conversely, if Λ is an isomorphism, then for any bilinear map $\varphi : M \times N \to Z$, the R-linear map $\Lambda^{-1}(\varphi)$ satisfies $\Lambda^{-1}(\varphi) \circ \tau = \varphi$, and it is the unique R-linear map with this propery. This shows that τ satisfies the universal property. $\square$

6.6 Remark. Let us look at proposition 6.5 from a physicists point of view. Here, the motivation for studying tensor products it to obtain an efficient formalism to handle multilinear maps.

Suppose we are given vector spaces U and V over a field K, where for example $K = \mathbb{R}$ or $K = \mathbb{C}$. We want to understand bilinear maps $\varphi : U \times V \to K$. With the use of a tensor product $\tau : U \times V \to T$, we can translate bilinear maps into K-linear maps. So for applications, one would not focus on the K-vector space T, but rather on the space of K-linear maps from T to K. By definition, this is just the dual vector space T^*.

So when physicists talk about tensors, they are thinking of elements in T^*, while we will be defining tensors as elements of the tensor product T, see 6.15 below. Since any finite dimensional vector space is isomorphic to its dual, the two interpretations are interchangeable.

6.7 Remark. In fact, proposition 6.5 can be strengthened as follows: a bilinear map $\tau : M \times N \to T$ is a tensor product, if and only if there exists an isomorphism η of functors from $\mathrm{Hom}_R(T, \bullet)$ to $L_R(M, N; \bullet)$, such that $\tau = \eta_T(\mathrm{id}_T)$. In categorical language, this is expressed by saying that the functor $L_R(M, N; \bullet)$ is *represented* by τ.

6.8 Definition. Let $\mathbf{F} : \mathcal{B} \to \mathcal{C}$ be a functor. A pair (U, u), with $U \in \mathrm{Ob}(\mathcal{B})$ and $u \in F(U)$ is called a *universal pair* for the functor $\mathbf{F}$, if it satisfies the following property:

$(*)$ for all objects $B \in \mathrm{Ob}(\mathcal{C})$ and all elements $c \in F(B)$, there exists a unique morphism $f \in \mathrm{Mor}_{\mathcal{B}}(U, B)$, such that $c = F(f)(u)$.

In this case, U is called a *universal object* and u is called a *universal element* for $\mathbf{F}$. The property $(*)$ is referred to as the *universal property* of u.

6.9 Example. Let a pair M and N of R-modules be given, and let $\tau : M \times N \to T$ be a tensor product. As functor $\mathbf{F}$, we consider $L_R(M, N; \bullet) : (R\text{-Mod}) \to (R\text{-Mod})$. Then, for any R-module Z, and all elements $\varphi \in L_R(M, N; Z)$, there exists a unique homomorphism $\tilde{\varphi} \in \operatorname{Hom}_R(T, Z)$, such that $\varphi = \tilde{\varphi}_*(\tau)$. Thus, by definition, the pair (T, τ) is a universal pair for $L_R(M, N; \bullet)$.

6.10 Proposition. *Let $\mathbf{F} : \mathcal{B} \to (\text{Set})$ be a functor, and let (U, u) be a pair with $U \in \operatorname{Ob}(\mathcal{B})$ and $u \in F(U)$. Then the following are equivalent:*

(1) the pair (U, u) is universal for $\mathbf{F}$;

(2) there exists an isomorphism of functors $\eta : \mathbf{Mor}(U, \bullet) \Rightarrow \mathbf{F}$ such that $\eta_U(\operatorname{id}_U) = u$.

Proof. (i) Suppose that η is an isomorphism from $\mathbf{Mor}(U, \bullet)$ to $\mathbf{F}$ which satisfies $\eta_U(\operatorname{id}_U) = u$. Then clearly $u \in F(U)$, so we only need to verify the universal property for u. Let $B \in \operatorname{Ob}(\mathcal{B})$, and let $c \in F(B)$ be an arbitrary element. Since $\eta_B : \operatorname{Mor}_{\mathcal{B}}(U, B) \to F(B)$ is an isomorphism, there exists a unique morphism $f \in \operatorname{Mor}_{\mathcal{B}}(U, B)$ such that $c = \eta_B(f)$. Consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Mor}_{\mathcal{B}}(U, U) \arrow[r, "\eta_U"] \arrow[d, "f_*"'] & F(U) \arrow[d, "F(f)"] \\
\operatorname{Mor}_{\mathcal{B}}(U, B) \arrow[r, "\eta_B"'] & F(B).
\end{tikzcd}$$

From this, we get $F(f)(u) = F(f) \circ \eta_U(\operatorname{id}_U) = \eta_B \circ f_*(\operatorname{id}_U) = \eta_B(f) = c$, as desired.

(ii) Conversely, suppose that (U, u) is universal for $\mathbf{F}$. Let $B \in \operatorname{Ob}(\mathcal{B})$. For an element $c \in F(B)$, let $f_c \in \operatorname{Mor}_{\mathcal{B}}(U; B)$ denote the unique morphism with $F(f_c)(u) = c$. We define a map

$$\begin{array}{rccc} \varrho_B : & F(B) & \to & \operatorname{Mor}_{\mathcal{B}}(U, B) \\ & c & \mapsto & f_c \end{array}$$

which is bijective by the assumption on the universality of (U, u). To prove that $\varrho := \{\varrho_B\}_{B \in \mathrm{Ob}(\mathcal{B})}$ is an isomorphism of functors, we need to show that for any morphism $g : B \to B'$ in $\mathcal{B}$ the diagram

$$\begin{tikzcd} F(B) \arrow[r, "\varrho_B"] \arrow[d, "F(g)"'] & \mathrm{Mor}_{\mathcal{B}}(U, B) \arrow[d, "g_*"] \\ F(B') \arrow[r, "\varrho_{B'}"'] & \mathrm{Mor}_{\mathcal{B}}(U, B'). \end{tikzcd}$$

is commutative. In other words, we need to verify the identity $\varrho_{B'} \circ F(g) = g_* \circ \varrho_B$ of morphisms from $F(B)$ to $\mathrm{Mor}_{\mathcal{B}}(U, B')$. So consider an element $c \in F(B)$. Evaluating the left hand side of the equation gives $\varrho_{B'} \circ F(g)(c) = f_{F(g)(c)}$. For the right hand side, we obtain $g_* \circ \varrho_B(c) = g_*(f_c) = g \circ f_c$. Now, applying the functor $\mathbf{F}$, and evaluating at u, we obtain

$$F(g_* \circ \varrho_B(c))(u) = F(g) \circ F(f_c)(u) = F(g)(c).$$

By definition, the last equation is equivalent to $g_* \circ \varrho_B(c) = f_{F(g)(c)}$. Comparing both sides finally shows that ϱ is a natural transformation. Since ϱ is invertible, defining η as its inverse concludes the proof. $\square$

6.11 Lemma. *Let $\tau : M \times N \to T$ be a tensor product of two R-modules M and N over R. Then $\mathrm{im}(\tau) = T$, and for any pair $f, g : T \to Z$ of R-module homomorphisms, the identity $f \circ \tau = g \circ \tau$ implies $f = g$.*

Proof. Let $T' := \mathrm{im}(\tau) = \mathrm{span}_R(\tau(M \times N))$. Let $i : T' \to T$ denote the inclusion homomorphism. By the definition of T', the bilinear map τ can obviously be written as a composition $\tau = i \circ \tau'$, where $\tau' : M \times N \to T'$ is given by $\tau'(m, n) := \tau(m, n)$ for all $(m, n) \in M \times N$. The map τ' is bilinear, so by the universal property of the tensor product, there exists a unique R-linear map $\tilde{\tau}' : T \to T'$ such that $\tau' = \tilde{\tau}' \circ \tau$. Consider the commutative diagram

$$\begin{tikzcd} M \times N \arrow[d, "\tau"'] \arrow[dr, "\tau'"] \arrow[drr, "\tau"] & & \\ T \arrow[r, "\tilde{\tau}'"] \arrow[rr, bend right, "\mathrm{id}_T"'] & T' \arrow[r, "i"] & T \end{tikzcd}$$

By the universal property of the tensor product, applied to the bilinear map τ itself, we must have $\mathrm{id}_T = i \circ \tilde{\tau}'$. In particular, i must be surjective, and hence $T' = T$.

Now let $f, g : T \to Z$ be homomorphisms of R-modules with $f \circ \tau = g \circ \tau$. Put $\gamma := f \circ \tau$. By lemma **??**, the map γ is bilinear. The universal property of the tensor product implies that the factorization of γ via τ is unique. Hence we must have $f = g$. □

As an application of lemma 6.11, we obtain a very useful criterion to determine, whether a given bilinear map is a tensor product. If is often referred to as the *weak universal property*.

6.12 Proposition. *Let $\tau : M \times N \to T$ be a bilinear map of R-modules, with $\mathrm{im}(\tau) = T$. Then τ is a tensor product of M and N over R, if and only if the following property holds:*

For any R-module Z, and any bilinear map $\varphi : M \times N \to Z$, there exists a homomorphism of R-modules $\tilde{\varphi} : T \to Z$, such that the diagram

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[d, "\tau"'] & Z \\ T \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

commutes.

Proof. Clearly, if τ is a tensor product, then the property of the proposition is satisfied by a homomorphism $\tilde{\varphi}$, which is even uniquely determined.

Conversely, suppose that the property of proposition 6.12 holds true. Then for any bilinear map $\varphi : M \times N \to Z$ of R-modules, there exists a homomorphism $\tilde{\varphi} : T \to Z$ with $\varphi = \tilde{\varphi} \circ \tau$. To prove, that τ is a tensor product, we need to show that $\tilde{\varphi}$ is uniquely determined by the condition $\varphi = \tilde{\varphi} \circ \tau$.

To do this, we consider a second homomorphism $\psi : T \to Z$ satisfying $\varphi = \psi \circ \tau$. Let $t \in T$. By assumption, $T = \mathrm{im}(\tau) = \mathrm{span}_R(\tau(M \times N))$. Hence there exists a number $k \in \mathbb{N}_{>0}$, together with elements $m_1, \dots, m_k \in M$, $n_1, \dots, n_k \in N$, and $a_1, \dots, a_k \in R$ such that $t = a_1\tau(m_1, n_1) + \dots + a_k\tau(m_k, n_k)$. Using the R-linearity of ψ and $\tilde{\varphi}$, we compute

$$\begin{aligned} \psi(t) &= a_1\psi(\tau(m_1, n_1)) + \dots + a_k\psi(\tau(m_k, n_k)) \\ &= a_1\varphi(m_1, n_1) + \dots + a_k\varphi(m_k, n_k) \\ &= a_1\tilde{\varphi}(\tau(m_1, n_1)) + \dots + a_k\tilde{\varphi}(\tau(m_k, n_k)) \\ &= \tilde{\varphi}(t). \end{aligned}$$

Hence $\psi = \tilde{\varphi}$ as claimed. □

6.13 Notation. For any pair of R modules M and N, for which a tensor product exists (which, in fact, is the case for all R-modules, as we will see in theorem 6.20 below), we choose once and for all one tensor product of M and N over R. This tensor product shall be denoted by

$$\tau : M \times N \to M \otimes_R N,$$

and for a pair $(m, n) \in M \times N$ we write $m \otimes n := \tau(m, n) \in M \otimes_R N$. By remark 6.3, any other tensor product of M and N over R is equal to $\tau : M \times N \to M \otimes_R N$ up to composition with a unique isomorphism.

6.14 Remark. By lemma 6.11, we have $M \otimes_R N = \mathrm{im}(\tau)$ for any tensor product $\tau : M \times N \to T$ over R. This is equivalent to saying that for any element $t \in M \otimes_R N$, there exists a number $k \in \mathbb{N}_{>0}$, together with elements $m_1, \ldots, m_k \in M$, $n_1, \ldots, n_k \in N$, and $a_1, \ldots, a_k \in R$ such that

$$t = a_1(m_1 \otimes n_1) + \ldots + a_k(m_k \otimes n_k).$$

6.15 Definition. Let $\tau : M \times N \to T$ be a tensor product of two R-modules M and N over R. An element $t \in M \otimes_R N$ is called a *tensor*. A tensor t is called *decomposable*, if there exists a pair $(m, n) \in M \times N$ such that $t = m \otimes n$.

6.16 Remark. Let $\tau : M \times N \to T$ be a tensor product of M and N over R. To get a better idea of the R-module structure on $M \otimes_R N$ in the notation of 6.13, we consider a tensor $t \in M \otimes_R N$ and an element $r \in R$.

By construction, there exists a $k \in \mathbb{N}$ and decomposing tensors $t_i = m_i \otimes n_i$, with $m_1, \ldots, m_k \in M$ and $n_1, \ldots, n_k \in N$, together with $a_1, \ldots, a_k \in R$ such that $t = a_1 \cdot m_1 \otimes n_1 + \ldots + a_k \cdot m_k \otimes n_k$.

Consider a decomposing tensor $t = m \otimes n$, with $m \in M$ and $n \in N$. Using the bilinearity of τ, we compute

$$r \cdot (m \otimes n) = r \cdot \tau(m, n) = \begin{cases} \tau(rm, n) = (rm) \otimes n \\ \tau(m, rn) = m \otimes (rn). \end{cases}$$

For future reference, we note the following rules for computation, for all $m, m' \in M$, $n, n' \in N$ and $r \in R$:

$$\begin{array}{lrcl} (1) & (rm) \otimes n & = & m \otimes (rn) \\ (2) & (m + m') \otimes n & = & m \otimes n + m' \otimes n \\ (3) & m \otimes (n + n') & = & m \otimes n + m \otimes n' \\ (4) & m \otimes 0 & = & 0 \\ (5) & 0 \otimes n & = & 0. \end{array}$$

The equalities (2) and (3) are seen analogously to our argument for the first equality. The last equality can be derived from (2) via $0 \otimes n = (0+0) \otimes n = 0 \otimes n + 0 \otimes n$, and analogously for $m \otimes 0 = 0$.

6.17 Example. Let M be an R-module. By the definition of a module, the operation $\lambda : R \times M \to M$ of R on M is bilinear. We claim that λ is a tensor product of R and M over R, and hence in particular

$$R \otimes_R M \cong M.$$

Obviously we have $\mathrm{im}(\lambda) = M$, so we may apply proposition 6.12 to prove the claim. Let Z be an arbitrary R-module, and let $\varphi : R \times M \to Z$ be bilinear. We define $\tilde{\varphi} : M \to Z$ by $\tilde{\varphi}(m) := \varphi(1, m)$ for $m \in M$. Note that $\tilde{\varphi}$ is R-linear, since φ is linear in the second argument by the definition of a bilinear map. For any $(r, m) \in R \times M$ we compute $\tilde{\varphi} \circ \lambda(r, m) = \tilde{\varphi}(rm) = \varphi(1, rm) = \varphi(r, m)$, so that $\tilde{\varphi} \circ \lambda = \varphi$.

6.18 Example. Let $(K, +, \cdot)$ be a field. Consider the K-vector space $V := K^n$ for some $n \in \mathbb{N}_{>0}$. Consider a second field $(L, +, \cdot)$, such that $K \subseteq L$. Note that L can be viewed as a vector space over K, too. We claim

$$L \otimes_K K^n \cong L^n.$$

As a special case, we obtain for $\mathbb{R} \subset \mathbb{C}$ the identification $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^n \cong \mathbb{C}^n$. To prove the claim, we consider the map

$$\begin{array}{rccc} \tau : & L \times K^n & \to & L^n \\ & (a, (v_1, \ldots, v_n)) & \mapsto & (av_1, \ldots, av_n) \end{array}$$

which is bilinear. Note that $\tau(L \times K^n) = L^n$, so we have in particular for the image of the bilinear map τ the equality $\mathrm{im}(\tau) = L^n$. Therefore, in order to prove that τ is a tensor product, it suffices to verify that τ satisfies the weak universal property of proposition 6.12.

Let Z be a K-vector space, and $\varphi : L \times K^n \to Z$ be a bilinear map. Note that $K^n \subseteq L^n$, and the standard basis $\{e_1, \ldots, e_n\} \subseteq K^n$ is simultaneously the standard basis for L^n. We define

$$\begin{array}{rccc} \tilde{\varphi} : & L^n & \to & Z \\ & (a_1, \ldots, a_n) & \mapsto & \varphi(a_1, e_1) + \ldots + \varphi(a_n, e_n). \end{array}$$

Using the bilinearity of φ with respect to K, we compute for $a \in L$ and $(v_1, \ldots, v_n) \in K^n$

$$\begin{aligned}
\tilde{\varphi} \circ \tau(a, (v_1, \ldots, v_n)) &= \tilde{\varphi}(av_1, \ldots, av_n) \\
&= \varphi(av_1, e_1) + \ldots + \varphi(av_n, e_n) \\
&= v_1\varphi(a, e_1) + \ldots + v_n\varphi(a, e_n) \\
&= \varphi(a, v_1e_1) + \ldots + \varphi(a, v_ne_n) \\
&= \varphi(a, v_1e_1 + \ldots + v_ne_n) \\
&= \varphi(a, (v_1, \ldots, v_n)).
\end{aligned}$$

Hence $\tilde{\varphi} \circ \tau = \varphi$, so proposition 6.12 implies that τ is a tensor product of L and K^n. By the uniqueness property of remark 6.3 we conclude $L^n \cong L \otimes_K K^n$.

6.19 Example. Consider the ring $(\mathbb{Z}, +, \cdot)$. Let $p, q \in \mathbb{N}$ be prime numbers with $p \neq q$. Let $M := \mathbb{Z}/p\mathbb{Z}$ and $N := \mathbb{Z}/q\mathbb{Z}$, considered as modules over $\mathbb{Z}$. For two integers $a, b \in \mathbb{Z}$, consider the decomposable tensor

$$\overline{a} \otimes \overline{b} \in \mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/q\mathbb{Z}.$$

Since p and q have no non-trivial common divisor, the equivalence class $\overline{q} \in \mathbb{Z}/p\mathbb{Z}$ is multiplicatively invertible. So there exists an $r \in \mathbb{Z}$ such that $\overline{q}\,\overline{r} = \overline{1}_{\mathbb{Z}/p\mathbb{Z}}$. Thus we compute with remark 6.16

$$\overline{a} \otimes \overline{b} = \overline{q}\,\overline{r}\,\overline{a} \otimes \overline{b} = q\,\overline{r}\,\overline{a} \otimes \overline{b} = \overline{r}\,\overline{a} \otimes q\,\overline{b} = \overline{r}\,\overline{a} \otimes \overline{q}\,\overline{b} = \overline{r}\,\overline{a} \otimes \overline{0} = 0.$$

Since any tensor product is generated by its decomposable tensors, we obtain the identity of $\mathbb{Z}$-modules

$$\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/q\mathbb{Z} = \{0\}.$$

6.20 Theorem. *Let $(R, +, \cdot)$ be a commutative ring with a multiplicative identity element. Then for any two R-modules M and N, there exists a tensor product $\tau : M \times N \to M \otimes_R N$.*

Proof. Let M and N be R-modules. We define $W := R\langle M \times N\rangle$ as the free R-module generated by the set $M \times N$. Note that the module structures of M and N get completely ignored. Thus

$$W := \left\{ \begin{array}{l} a_1 \cdot (m_1, n_1) + \ldots + a_k \cdot (m_k, n_k) : \\ k \in \mathbb{N},\ a_1, \ldots, a_k \in R,\ m_1, \ldots, m_k \in M,\ n_1, \ldots, n_k \in N \end{array} \right\}$$

is the set of all finite formal sums of pairs in $M \times N$ with coefficients in R. As usual, for $k = 0$, we denote the empty sum by 0. We define a submodule of W by

$$W_0 := \mathrm{span}_R \left\{ \begin{array}{l} 1 \cdot (r_1 m_1 + r_2 m_2, n) - r_1 \cdot (m_1, n) - r_2 \cdot (m_2, n), \\ 1 \cdot (m, s_1 n_1 + s_2 n_2) - s_1 \cdot (m, n_1) - s_2 \cdot (m, n_2) : \\ r_1, r_2, s_1, s_2 \in R,\ m_1, m_2 \in M,\ n_1, n_2 \in N \end{array} \right\}$$

Consider the R-module quotient $T := W/W_0$, and denote the canonical quotient homomorphism by $\pi : W \to T$. Let $i : M \times N \to W$ denote the natural inclusion map. We claim that a tensor product of M and N over R is given by

$$\begin{array}{rccc} \tau := \pi \circ i : & M \times N & \to & T \\ & (m, n) & \mapsto & [1 \cdot (m, n)] \end{array}$$

Indeed, for elements $r, s \in R$, $m, m' \in M$ and $n \in N$, we compute

$$\begin{aligned} \tau(rm + sm', n) &= [1 \cdot (rm + sm', n)] \\ &= [1 \cdot (rm + sm', n)] - [0] \\ &= [1 \cdot (rm + sm', n)] - [(1 \cdot (rm + sm', n) - r \cdot (m, n) - s \cdot (m', n))] \\ &= [r \cdot (m, n) + s \cdot (m', n)] \\ &= r[1 \cdot (m, n)] + s[1 \cdot (m', n)] \\ &= r\,\tau(m, n) + s\,\tau(m', n) \end{aligned}$$

An analogous computation for the second argument shows that τ is a bilinear map. The equality $\mathrm{im}(\tau) = T$ is clear from the construction of τ. Hence to prove the claim it suffices to prove that τ satisfies the weak universal property of proposition 6.12.

Let Z be an R-module, together with a bilinear map $\varphi : M \times N \to Z$. We define an R-linear map on the generators of W by

$$\begin{array}{rccc} \Phi : & W & \to & Z \\ & 1 \cdot (m, n) & \mapsto & \varphi(m, n) \end{array}$$

and extend R-linearly. We claim: $W_0 \subseteq \ker(\Phi)$. Indeed, consider a generator $w := 1 \cdot (r_1 m_1 + r_2 m_2, n) - r_1 \cdot (m_1, n) - r_2 \cdot (m_2, n)$ of W_0. Then we compute from the definition of Φ:

$$\begin{aligned} \Phi(w) &= \varphi(r_1 m_1 + r_2 m_2, n) - r_1 \varphi(m_1, n) - r_2 \varphi(m_2, n) \\ &= r_1 \varphi(m_1, n) + r_2 \varphi(m_2, n) - r_1 \varphi(m_1, n) - r_2 \varphi(m_2, n) \\ &= 0, \end{aligned}$$

using the bilinearity of φ. Now the universal property of the quotient module implies the existence of a homomorphism $\tilde{\varphi} : W/W_0 \to Z$ of R-modules, such that $\tilde{\varphi} \circ \pi = \Phi$. Consider the diagram

$$\begin{tikzcd} M \times N \arrow[r, "i"'] \arrow[rr, bend left, "\varphi"] \arrow[dr, "\tau"'] & W \arrow[r, "\Phi"'] \arrow[d, "\pi"] & Z \\ & W/W_0 \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

We compute from the definition of $\tau := \pi \circ i$ the identities

$$\tilde{\varphi} \circ \tau = \tilde{\varphi} \circ \pi \circ i = \Phi \circ i = \varphi$$

giving the desired factorization $\varphi = \tilde{\varphi} \circ \tau$ via the R-module $T = W/W_0$. □

6.21 Remark. (The functorial property of the tensor product)

Let an R-module M be given. For any two R-modules N and N' let $\tau : M \times N \to M \otimes N$ and $\tau' : M \times N' \to M \otimes N'$ denote the respective tensor products with M. Consider a homomorphism $\alpha : N \to N'$ of R-modules. Clearly, the product map

$$\mathrm{id}_M \times \alpha : \quad M \times N \to M \times N'$$

is R-linear. The composition $\varrho := \tau' \circ (\mathrm{id}_M \times \alpha) : \ M \times N \to M \otimes N'$ is bilinear, as one easily verifies. By the universal property of the tensor product τ, there exists a unique R-linear map $\tilde{\varrho}$ which makes the following diagram commutative:

$$\begin{tikzcd} M \times N \arrow[r, "\mathrm{id}_M \times \alpha"] \arrow[d, "\tau"'] \arrow[dr, "\varrho"] & M \times N' \arrow[d, "\tau'"] \\ M \otimes N \arrow[r, "\tilde{\varrho}"'] & M \otimes N' \end{tikzcd}$$

This unique homomorphism shall be denoted by $\mathrm{id}_M \otimes \alpha := \tilde{\varrho}$. It satisfies the identity

$$(\mathrm{id}_M \otimes \alpha) \circ \tau = \tau' \circ (\mathrm{id}_M \times \alpha).$$

For a decomposing tensor $m \otimes n \in M \otimes_R N$, with $m \in M$ and $n \in N$, we compute

$$\mathrm{id}_M \otimes \alpha(m \otimes n) = (\mathrm{id}_M \otimes \alpha) \circ \tau(m, n) = \tau' \circ (\mathrm{id}_M \times \alpha)(m, n) = m \otimes \alpha(m).$$

From this, we obtain for two additional homomorphisms $\alpha' : N \to N'$ and $\beta : N' \to N''$ the identities

$$\begin{array}{rcl} \mathrm{id}_M \otimes (\alpha + \alpha') & = & \mathrm{id}_M \otimes \alpha + \mathrm{id}_M \otimes \alpha' \\ \mathrm{id}_M \otimes (\beta \circ \alpha) & = & (\mathrm{id}_M \otimes \beta) \circ (\mathrm{id}_M \otimes \alpha). \end{array}$$

Summing things up, we obtain for a given R-module M a functor

$$\begin{array}{rccc} M \otimes_R \bullet : & (R\text{-Mod}) & \to & (R\text{-Mod}) \\ & N & \mapsto & M \otimes_R N \\ & N \stackrel{\alpha}{\to} N' & \mapsto & M \otimes_R N \stackrel{\mathrm{id}_M \otimes \alpha}{\longrightarrow} M \otimes_R N' \end{array}$$

which is covariant and additive.

6.22 Remark. In fact, one can prove that the functor $M \otimes_R \bullet$ is exact on the right. In particular, for any surjective homomorphism $\alpha : N \to N'$ of R-modules, the induced morphism $\mathrm{id}_M \otimes \alpha : M \otimes N \to M \otimes N'$ is surjective, too.

6.23 Remark. Analogously as in remark 6.21, a given R-module N determines a functor $\bullet \otimes N$ by assigning to an R-module M the R-module $M \otimes_R N$. Consider a pair of homomorphisms of R-modules $\beta : M \to M'$ and $\alpha : N \to N'$. We claim that the diagram

$$\begin{tikzcd} M \otimes N \arrow[r, "\mathrm{id}_M \otimes \alpha"] \arrow[d, "\beta \otimes \mathrm{id}_N"'] & M \otimes_R N' \arrow[d, "\beta \otimes \mathrm{id}_{N'}"] \\ M' \otimes_R N \arrow[r, "\mathrm{id}_{M'} \otimes \alpha"'] & M' \otimes_R N'. \end{tikzcd}$$

is commutative. Indeed, consider the tensor product $\tau : M \times N \to M \otimes_R N$. If we compose τ with an R-linear map, the composition is still bilinear by remark **??**. Hence we obtain two bilinear maps $\varphi_1, \varphi_2 : M \times N \to M' \otimes_R N'$ by defining $\varphi_1 := (\beta \otimes \mathrm{id}_{N'}) \circ (\mathrm{id}_M \otimes \alpha) \circ \tau$ and $\varphi_2 := (\mathrm{id}_{M'} \otimes \alpha) \circ (\beta \otimes \mathrm{id}_N) \circ \tau$. For any pair $(m, n) \in M \times N$, we compute directly from the definitions $\varphi_1(m, n) = \varphi_2(m, n)$, so that $\varphi_1 = \varphi_2$. By the universal property of the tensor product τ, the bilinear map φ_1 factorizes in unique way as $\varphi_1 = \tilde{\varphi}_1 \circ \tau$. Therefore we must have $\tilde{\varphi}_1 = (\beta \otimes \mathrm{id}_{N'}) \circ (\mathrm{id}_M \otimes \alpha) = (\beta \otimes \mathrm{id}_{N'}) \circ (\mathrm{id}_M \otimes \alpha)$.

For this unique map, and its evaluation on decomposable elements, we write

$$\begin{array}{rccc} \beta \otimes \alpha : & M \otimes_R N & \to & M' \otimes_R N' \\ & m \otimes n & \mapsto & \beta(m) \otimes \alpha(n). \end{array}$$

In categorical terms, this gives rise to a bifunctor

$$\otimes : \quad (R\text{-Mod}) \times (R\text{-Mod}) \to (R\text{-Mod}).$$

Be warned, however, that the above notation of $\beta \otimes \alpha$ has its issues: In the way, we introduced it, it denotes a homomorphism $\beta \otimes \alpha \in \mathrm{Hom}_R(M \otimes_R N, M' \otimes_R N')$. At the same time, α and β are elements of their respective R-modules of homomorphisms, so there is a well-defined element $\beta \otimes \alpha \in \mathrm{Hom}_R(M, N) \otimes_R \mathrm{Hom}_R(M', N')$. In general, these two interpretations of the symbol $\beta \otimes \alpha$ are not the same, and not even corresponding to each other under an isomorphism, since without some extra hypothesis we have

$$\mathrm{Hom}_R(M, N) \otimes_R \mathrm{Hom}_R(M', N') \not\cong \mathrm{Hom}_R(M \otimes_R N, M' \otimes_R N').$$